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More comments on string theory on AdS_3

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Abstract: We clarify a number of issues regarding the worldsheet and spacetime descriptions of string propagation on AdS_3. We construct the vertex operators of spacetime current algebra and spacetime (super) Virasoro generators in the full interacting SL(2) WZW theory and study their Ward identities. We also explain the relation between the analysis in this note and some recent work on this subject.

Keywords: Sigma Models, Conformal Field Models in String Theory, Black Holes in String Theory.

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Contents

1. Introduction

String theory on AdS_3 has many applications. It is related to the seminal work of Strominger and Vafa on five dimensional black holes [1]. It also provides an example of Maldacena's AdS/CFT correspondence [2]. In the context of this correspondence this example is special because the "boundary" (or spacetime) CFT is two dimensional, and therefore it has an infinite dimensional conformal algebra. This Virasoro algebra was first found by Brown and Henneaux [3], who studied gravity on AdS_3. The AdS_3 example is furthermore special since it is amenable to a worldsheet string theory description going beyond the (super) gravity approximation. The spacetime Virasoro symmetry of string theory on AdS_3 was exhibited in [4], where it was also shown that string excitations form representations of this symmetry, in both bosonic and supersymmetric string theories. The work of [4] in the bosonic case was later extended in [5]. The superstring was further discussed by [6, 7] and others. The results of [4] allow one to analyze vacua with Neveu-Schwarz (NS) background fields in the NSR formalism. Ramond-Ramond (RR) backgrounds were discussed in [8, 11].

One important lesson from these stringy analyses is that (some of) the observables in the theory correspond to fundamental string excitations and are thus described by standard vertex operators in the BRST cohomology. Their worldsheet correlation functions, which in flat backgrounds are usually interpreted as S-matrix elements in target space, are interpreted here as Green's functions of operators in the spacetime CFT [12, 13].

In this note we will further extend the work of [4]. We will mainly use worldsheet techniques, but in this section we start with a target space analysis, which provides useful guidance to some aspects of the physics.

Consider three dimensional gravity with a negative cosmological constant. The lagrangean is

$$\frac{1}{\hbar}\sqrt{-g}\left(\frac{2}{k}+R\right). \tag{1.1}$$

The vacuum solution of its classical equations of motion is AdS_3. Three dimensional gravity does not have propagating degrees of freedom. However, by a careful analysis of the boundary conditions and the gauge transformations one should mod out by, Brown and Henneaux showed [3] that this theory has two Virasoro symmetries with central charge

$$c=\frac{24\pi\sqrt{k}}{\hbar}. \tag{1.2}$$

The $\hbar$ in the denominator shows that the effect is classical and visible at the level of Poisson brackets (as in the computation of [3]). It is often said that this Virasoro algebra lives at the boundary of the AdS spacetime. More precisely, the statement is that most of the degrees of freedom of the graviton are pure gauge, with a remnant which can be gauge transformed to infinity. In perturbative string theory, choosing the Lorentz covariant conformal gauge on the worldsheet leads to Landau gauge in target space. In this gauge the physical degrees of freedom of the graviton are not supported only at the boundary but in the entire AdS_3.

Following Horowitz and Welch [14], (1.1) can be extended (as in string theory) by adding the dilaton and the NS $H=dB$ field

$$e^{-2\phi}\sqrt{-g}\left(\frac{4}{k}+R+4(\nabla\phi)^2-\frac{1}{12}H_{\mu\nu\rho}^2\right), \tag{1.3}$$

where $\hbar$ was absorbed into a shift of ϕ. Solving the equations of motion we find (as usual in string theory) that all the fields are determined except the zero mode of the dilaton, which is an integration constant

$$\begin{aligned} e^{-2\phi} &= \frac{1}{\hbar} \\ g_{\mu\nu} &= k g_{\mu\nu}^{(0)} \\ H &= \frac{1}{\sqrt{k}}\epsilon = kH^{(0)}, \end{aligned} \tag{1.4}$$

where ϵ is the volume form and $g^{(0)}_{\mu\nu}$ and $H^{(0)}$ are independent of k. As above, the Brown and Henneaux analysis leads to a Virasoro algebra with the central charge (1.2).

The spectrum of the theory (1.3) includes a single particle, the dilaton, whose mass is of order $\frac{1}{\sqrt{k}}$. The dilaton is sometimes referred to as a fixed scalar. Despite the fact that the dilaton is massive, its zero mode $\hbar$ is an arbitrary integration constant. In other words, the string coupling constant is arbitrary even though there is no massless field which changes its value.[1]

The value of H can be interpreted as arising from an electric source for H of strength

$$p = \frac{4\pi}{\hbar\sqrt{k}}\,, \tag{1.5}$$

with coupling $\frac{p}{4\pi}\oint B$ at infinity. Therefore, we can replace the arbitrary integration constant $\hbar$ by the constant p. Three dimensional gravity with non-zero H is analogous to QED_2 with a constant background electric field, $E = \hbar\theta/2\pi$, which can be thought of as resulting from an electric charge $\theta/2\pi$ at infinity. QED_2 is periodic in θ; this is a result of spontaneous creation of particles from the bulk and screening of the source at infinity. Because of the constant negative curvature in our problem, the volume is proportional to the area, and there is no gain in bulk energy while losing surface energy in the nucleation process. Hence, there is no spontaneous nucleation, and we do not expect periodicity in p. Although spontaneous nucleation does not happen, a long string can be created with a finite amount of energy [16]. The nucleation process and the physics of the long string are discussed in detail in [17]. We will return to the long string in section 7.

We will see later that, at least in some situations, the integration constant (1.5) is quantized due to non-perturbative effects. The number p is then interpreted as the number of fundamental strings creating the AdS background. It is natural to conjecture that all such arbitrary integration constants which are not associated with massless fields are quantized in string theory.

We now add to (1.3) gauge fields of a group G, like those in the NS sector in string theory. They are associated with a $\widehat{G}$ current algebra on the worldsheet; the level of $\widehat{G}$ is an integer k_G. The target space lagrangean is

$$e^{-2\phi}\sqrt{-g}\left(\frac{4}{k} + R + 4(\nabla\phi)^2 - \frac{1}{12}H^2_{\mu\nu\rho} + k_G F^2_{\mu\nu}\right), \tag{1.6}$$

with

$$H = dB + k_G\omega\,, \tag{1.7}$$

where $\omega = \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ is the Chern-Simons form of the gauge fields. The solution of the equations of motion is as in (1.4). Expanding the lagrangean (1.6)

[1]This situation is similar to the quasi-crystalline compactification to two dimensions [15].

around the classical solution we find the Chern-Simons interaction

$$\frac{k_G p}{4\pi}\omega\,, \tag{1.8}$$

which leads to masses of order $1/\sqrt{k}$ for the gauge fields.

The Chern-Simons interaction also leads to current algebra on the boundary [18] with level

$$k_G^{(st)} = k_G p\,. \tag{1.9}$$

The result (1.9) was derived in [4] using worldsheet techniques. We now see that the spacetime analysis leads to the same conclusion.

As in the case of the Virasoro algebra discussed above, the spacetime current algebra can be gauge transformed to the boundary of AdS_3. In string theory in conformal gauge we are naturally led to the Landau gauge $\partial^\mu A_\mu = 0$. The remaining gauge freedom corresponds to gauge transformations with a harmonic gauge parameter λ. The physical degrees of freedom are massive spin one particles and various modes which are almost pure gauge. These are modes with harmonic λ which do not vanish at infinity. These modes lead to the current algebra. Their profile is non-zero in the bulk of the space-time, as we will see below, when we construct the vertex operators of these modes.

It is now clear that p must be quantized. This follows either from unitarity of the spacetime current algebra or from the invariance of the lagrangean under large gauge transformations. The quantization of p and its interpretation in terms of the number of strings creating the background are in accord with the general expectation that the only consistent backgrounds in string theory are those which can occur due to sources which are themselves dynamical objects in the theory.[2] This is another reason to support the conjecture above that the corresponding integration constants in string theory are quantized. It is also clear that the quantization of p is non-perturbative in the coupling, and cannot be seen at any finite order in the $\hbar \sim 1/p$ genus expansion of string theory.

Note that the discussion so far is completely general. It holds for any theory of gravity with negative cosmological constant which is described at low energies by the appropriate lagrangean, (1.1), (1.3) or (1.6). Similarly, much of the rest of this note is true for any vacuum of string theory that includes an AdS_3 factor.

We now add more structure to (1.6) by adding to it supersymmetry. In particular, we are interested in the compactification of the type II theory on $T^4 \times S^3$ to three dimensions. The bosonic lagrangean includes the terms (1.6) for $G = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_L^4 \times \mathrm{U}(1)_R^4$ gauge fields coming from Kaluza-Klein reduction of the ten dimensional metric and NS B field. The SU(2) factors arise from the S^3 and

[2]Our problem is analogous to the problem of the massive IIA theory in 10 dimensions analyzed by Polchinski and Strominger [19]. They showed that the source at infinity is quantized in units of the $D8$-branes which are present in the theory.

the U(1) factors from the T^4. The Chern-Simons terms (1.8) for the gauge groups labeled by L and R have opposite signs. All the massive gauge bosons have mass of order $1/\sqrt{k}$. Those with one sign of the Chern-Simons term have spin plus one and the others have spin minus one. The spacetime theory includes a $\widehat{G}$ current algebra. The level (1.9) of the $\mathrm{SU}(2)_R$ factor is $k^{(st)}_{\mathrm{SU}(2)_R} = kp$ (k and p are defined as before by (1.5), (1.6), while that of the U(1) factors is $k^{(st)}_{\mathrm{U}(1)_R} = p$. $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)^4_L$ have the same levels as $\mathrm{SU}(2)_R$ and $\mathrm{U}(1)^4_R$ but the opposite orientations.

Additional gauge fields arise from the RR sector. These are most easily analyzed in the IIA language. In ten dimensions the RR gauge fields are a one form A and a three form C. The relevant terms in the ten dimensional lagrangean are

$$\frac{1}{2}(dA)^2 + \frac{1}{4!}(dC + H \wedge A)^2 + C \wedge dC \wedge H\,. \qquad (1.10)$$

For simplicity we consider a square T^4 with equal sides $V^{\frac{1}{4}}$. We denote the directions along the T^4 by i, j, k, l, and the three non-compact dimensions by μ. The light vector fields in three dimensions (which are s-waves on the sphere and constant on the T^4) are: $A = A_\mu$, six fields $A_{\mu,ij} = C_{ij\mu}$, and $\widetilde{A}_\mu = \epsilon^{ijkl}\widetilde{C}_{ijkl\mu}$ ($\widetilde{C}$ is a five form gauge field, which is dual in ten dimensions to C). After dimensional reduction to three dimensions and a duality transformation on C we find

$$V(dA)^2 + \frac{1}{V}(d\widetilde{A})^2 + (dA_{ij})^2 + kA \wedge d\widetilde{A} + k\epsilon^{ijkl}A_{ij} \wedge dA_{kl}\,, \qquad (1.11)$$

where the terms proportional to k arise from using the expectation value of H on S^3.

In order to find the spectrum of the theory we should diagonalize the mass matrix. We find eight massive spin one particles with mass of order $1/\sqrt{k}$. Four of them (one linear combination of A and $\widetilde{A}$, and the three A_{ij} which are self-dual on T^4) have spin plus one, and the other four have spin minus one. We also find $\mathrm{U}(1)^4_L$ and $\mathrm{U}(1)^4_R$ current algebras whose level is k, the coefficient of the Chern-Simons terms in (1.11).

The full current algebra of the model thus includes a $\mathrm{U}(1)^8_L \times \mathrm{U}(1)^8_R$ current algebra, associated with a lattice of signature $(8, 8)$. In the background we study, which is purely NS, this lattice is the sum of two signature $(4, 4)$ lattices; one for the NS fields and the other for the RR fields. The corresponding levels of the current algebras are p and k.

We can also use the above comments to extend the analysis of [7] and study the heterotic string on $AdS_3 \times S^3 \times T^4$. At generic points in the Narain moduli space the gauge group is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)^4_L \times \mathrm{U}(1)^{20}_R$, where all the gauge fields arise from the NS sector. An analysis of the three dimensional effective action as above shows that the levels of the two SU(2) factors are kp and $-(k-2)p$, and the $\mathrm{U}(1)^{24}$ has level p and is related to a signature $(4, 20)$ lattice. This theory is S-dual to the type IIA theory on $AdS_3 \times S^3 \times K3$, also with NS background field. The symmetry is again $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_L^4 \times \mathrm{U}(1)_R^{20}$, but here the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ is from the NS sector and has level[3] $(kp, -k(p-2))$, while the $\mathrm{U}(1)_L^4 \times \mathrm{U}(1)_R^{20}$ is from the RR sector, still with a signature $(4,20)$ lattice but with level k. These conclusions are consistent with S duality, which exchanges five branes and one branes, i.e. k and p.

In the rest of this note we will develop a worldsheet description of string theory on AdS_3, which is consistent with the general picture described in this section and makes it more precise. We will construct the vertex operators of the affine Lie algebra and Virasoro generators described above, in the exact worldsheet theory, and show that their correlation functions satisfy the Ward identities of two dimensional CFT.

One important lesson we will learn concerns the origin of the singularities in correlation functions in the spacetime CFT. Such singularites occur when two operators approach each other on the boundary. The locations of the operators on the boundary are parameters labeling the corresponding vertex operators in the first quantized theory. As these parameters are varied, singularities can occur either because the worldsheet functional integral over the non-compact target space diverges, or because the integral over the location of vertex operators on the worldsheet diverges. The analysis in [4] shows that the singularities occur from the region where vertex operators approach each other on the worldsheet. Thus, short distances on the worldsheet are mapped by the AdS/CFT correspondence to short distances on the boundary. The discussion in [5] and its extension below further shows that the above divergences in the boundary theory arise because the target space is non-compact and are dominated by the asymptotic region of the target space. Thus, short distance behavior on the worldsheet and on the boundary is related to long distance behavior in the target space gravity theory. All this is consistent with — and nicely demonstrates — the well known relation between short distance physics on the worldsheet and long distance physics in target space, which is generally true in string theory, and the UV/IR relation of [20], which relates the short distance regime of the spacetime CFT to the asymptotic region of the dual string theory target space.

We start in section 2 by describing classical and quantum CFT on AdS_3. In section 3 we use this description to construct vertex operators corresponding to the generators of a spacetime affine Lie algebra associated with a current algebra on the worldsheet of the string. In section 4 we prove that these vertex operators satisfy the correct spacetime current algebra Ward identities and find the operator that plays the role of the central extension in this current algebra. In section 5 we study this central extension and compute the level of the spacetime current algebra.

[3] The difference of $2k$ between the levels of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ in this case is due to a one string loop correction. Our formalism should allow one to compute these corrections by studying the two point functions of various currents on the (worldsheet) torus. Since the asymmetry in question is due to an anomaly, one should compute the contribution of the odd $(++)$ spin structure to such two point functions. Calculating this one string loop shift would provide interesting tests of our formalism and of heterotic – type IIA duality.

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In section 6 we generalize the construction to the Virasoro case. We find the vertex operator of the stress tensor of the spacetime theory and comment on some of its properties. In section 7 we explain the relation of the present construction to that of [4]. Section 8 contains some comments on the generalization of our results to the superstring. We summarize our results in section 9.

2. Conformal field theory on AdS_3

In this section we describe some properties of (worldsheet) CFT on AdS_3, which will be useful later in the analysis of string theory in this background. We focus on the Euclidean version of AdS_3, the non-compact manifold $H_3^+ = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. For some early work on CFT and string theory on AdS_3 see e.g. [21].

2.1 Classical CFT on AdS_3

H_3^+ has constant negative curvature $\Lambda = -2/k$ (in string units). It is described by the metric

$$ds^2 = k\left(d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}\right), \tag{2.1}$$

where $(\phi, \gamma, \bar{\gamma})$ are coordinates on H_3^+, with $\phi^* = \phi$, $\gamma^* = \bar{\gamma}$. To define a CFT with this target space one has to turn on additional σ-model couplings. One possibility is to turn on a Neveu-Schwarz $B_{\mu\nu}$ field, which leads to the worldsheet lagrangean

$$L = k\left(\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}\right). \tag{2.2}$$

The CFT described by (2.2) has an infinite affine $\mathrm{SL}(2) \times \overline{\mathrm{SL}(2)}$ symmetry which, as we will see below, is very helpful in studying the theory. The global part of the algebra is generated by

$$\begin{aligned}
J_0^- &= \partial_\gamma\,, \qquad J_0^3 = \gamma\partial_\gamma - \frac{1}{2}\partial_\phi\,, \qquad J_0^+ = \gamma^2\partial_\gamma - \gamma\partial_\phi - e^{-2\phi}\partial_{\bar{\gamma}} \\
\bar{J}_0^- &= \partial_{\bar{\gamma}}\,, \qquad \bar{J}_0^3 = \bar{\gamma}\partial_{\bar{\gamma}} - \frac{1}{2}\partial_\phi\,, \qquad \bar{J}_0^+ = \bar{\gamma}^2\partial_{\bar{\gamma}} - \bar{\gamma}\partial_\phi - e^{-2\phi}\partial_\gamma.
\end{aligned} \tag{2.3}$$

A large class of observables in this theory is obtained by studying functions on H_3^+. Such functions can be decomposed in terms of representations of $\mathrm{SL}(2) \times \overline{\mathrm{SL}(2)}$. A convenient tool for performing this decomposition was proposed in [22] (and further developed in [23]; see also [24]). One introduces an auxiliary complex variable $(x, \bar{x})$ and uses the standard representation of the $\mathrm{SL}(2) \times \overline{\mathrm{SL}(2)}$ Lie algebra as the differential operators

$$\begin{aligned}
J_0^- &= -\partial_x\,, \qquad J_0^3 = -(x\partial_x + h)\,, \qquad J_0^+ = -(x^2\partial_x + 2hx) \\
\bar{J}_0^- &= -\partial_{\bar{x}}\,, \qquad \bar{J}_0^3 = -(\bar{x}\partial_{\bar{x}} + \bar{h})\,, \qquad \bar{J}_0^+ = -(\bar{x}^2\partial_{\bar{x}} + 2\bar{h}\bar{x}),
\end{aligned} \tag{2.4}$$

where h is related to the SL(2) "spin" of the representation[4] j by $h = j + 1$. We will refer to the auxiliary space labeled by $(x, \bar{x})$ as "spacetime" below, to be distinguished from the three dimensional target space of the σ-model, H_3^+, labeled by $(\phi, \gamma, \bar{\gamma})$. $(x, \bar{x})$ label the space on which the "dual" (or spacetime) CFT lives.

One is looking for functions $f_{h,\bar{h}}(\phi, \gamma, \bar{\gamma}; x, \bar{x})$ that transform in the spin $(j, \bar{j}) = (h-1, \bar{h}-1)$ representation of $\mathrm{SL}(2) \times \overline{\mathrm{SL}(2)}$. The functions $f_{h,\bar{h}}$ satisfy the differential equations

$$\begin{aligned}
[J_0^-, f_{h,\bar{h}}] &= -\partial_x f_{h,\bar{h}} = \partial_\gamma f_{h,\bar{h}} \\
[\bar{J}_0^-, f_{h,\bar{h}}] &= -\partial_{\bar{x}} f_{h,\bar{h}} = \partial_{\bar{\gamma}} f_{h,\bar{h}} \\
[J_0^3, f_{h,\bar{h}}] &= -(x\partial_x + h) f_{h,\bar{h}} = \left(\gamma\partial_\gamma - \frac{1}{2}\partial_\phi\right) f_{h,\bar{h}} \\
[\bar{J}_0^3, f_{h,\bar{h}}] &= -(\bar{x}\partial_{\bar{x}} + \bar{h}) f_{h,\bar{h}} = \left(\bar{\gamma}\partial_{\bar{\gamma}} - \frac{1}{2}\partial_\phi\right) f_{h,\bar{h}} \\
[J_0^+, f_{h,\bar{h}}] &= -(x^2\partial_x + 2hx) f_{h,\bar{h}} = \left(\gamma^2\partial_\gamma - \gamma\partial_\phi - e^{-2\phi}\partial_{\bar{\gamma}}\right) f_{h,\bar{h}} \\
[\bar{J}_0^+, f_{h,\bar{h}}] &= -(\bar{x}^2\partial_{\bar{x}} + 2\bar{h}\bar{x}) f_{h,\bar{h}} = \left(\bar{\gamma}^2\partial_{\bar{\gamma}} - \bar{\gamma}\partial_\phi - e^{-2\phi}\partial_\gamma\right) f_{h,\bar{h}}.
\end{aligned} \tag{2.5}$$

The brackets $[\cdots]$ in these equations are Poisson brackets since we are in the classical theory. The most general solution of the first four equations in (2.5) is

$$f_{h,\bar{h}} = \frac{1}{(\gamma - x)^h (\bar{\gamma} - \bar{x})^{\bar{h}}} H_{h,\bar{h}}\left((\gamma - x)(\bar{\gamma} - \bar{x})e^{2\phi}\right), \tag{2.6}$$

with $H_{h,\bar{h}}$ an arbitrary function. The last two equations in (2.5) have solutions *only* for $h = \bar{h}$, in which case one finds (in a convenient normalization)

$$f_{h,h} \equiv \Phi_h = \frac{1}{\pi}\left(\frac{1}{(\gamma - x)(\bar{\gamma} - \bar{x})e^\phi + e^{-\phi}}\right)^{2h}. \tag{2.7}$$

Since Φ_h is an eigenfunction of the Laplacian in the bulk, it is the propagator of a particle with mass squared $h(h-1)/k$ from the boundary point x to the bulk point $(\phi, \gamma, \bar{\gamma})$. Note also that (2.5)–(2.7) imply that one can think of Φ_h as components of a tensor of weight (h, h) in spacetime, $\Phi_h(x, \bar{x}) dx^h d\bar{x}^h$. This tensor is non-singular for finite $(x, \bar{x})$ (for fixed, finite $(\phi, \gamma, \bar{\gamma})$). It is also regular at $x \to \infty$; to exhibit that, redefine $x' = 1/x$, and use the transformation $\Phi'_h(x', \bar{x}') dx'^h d\bar{x}'^h = \Phi_h(x, \bar{x}) dx^h d\bar{x}^h$.

We can expand Φ_h around $\phi \approx \infty$

$$\begin{aligned}
\Phi_h &= \frac{1}{\pi}\left(\frac{1}{|\gamma - x|^2 e^\phi + e^{-\phi}}\right)^{2h} \\
&= \frac{1}{2h-1} e^{2(h-1)\phi} \delta^2(\gamma - x) + \mathcal{O}(e^{2(h-2)\phi}) + \frac{e^{-2h\phi}}{\pi|\gamma - x|^{4h}} + \mathcal{O}(e^{-2(h+1)\phi}).
\end{aligned} \tag{2.8}$$

[4]The value of the quadratic Casimir in the representation is $j(j+1)$.

For generic h the expansion (2.8) can be naturally separated into two independent series. One series includes the leading term in the $\phi \to \infty$ limit,[5] $e^{2(h-1)\phi}\delta^2(\gamma - x)$, and an infinite series of corrections of the form $e^{2(h-n-1)\phi}\partial_x^n\bar{\partial}_{\bar{x}}^n\delta^2(\gamma - x)$. This series is present only in the vicinity of $\gamma = x$. The second series starts with the dominant term for generic γ, $e^{-2h\phi}/|\gamma - x|^{4h}$, and includes corrections that are down by $e^{-2n\phi}$ with $n \in \mathbb{Z}_+$.

The situation here is similar to that in Liouville theory [25]. The analogs of the wavefunctions Φ_h (2.7) in that case are the exact minisuperspace wavefunctions, the modified Bessel functions K_ν. The operators (2.7) with real h are the analogs of microscopic operators in Liouville theory. The analog of the expansion (2.8) is the expression of K_ν as a linear combination of $I_{\pm\nu}$. As in (2.8), for generic ν expanding $I_{\pm\nu}$ around $\phi \to \infty$ gives rise to two infinite series that do not mix. For integer ν the two series mix and interesting "resonances" occur. Analogous phenomena exist in our case; the two series in (2.8) mix when $h \in \mathbb{Z}/2$.

As we approach $h = 1/2$ in (2.7), the behavior of Φ_h changes:

1. For $h > \frac{1}{2}$ (and $\gamma \simeq x$) the leading term at large ϕ is $e^{2(h-1)\phi}\delta^2(\gamma - x)$.

2. For $h = \frac{1}{2}$ we have a "resonance" ($e^{2(h-1)\phi} = e^{-2h\phi}$) and the leading term in the expansion of (2.8) is $2\phi e^{-\phi}\delta^2(\gamma - x)$. This is consistent with the fact that the coefficient of $e^{2(h-1)\phi}\delta^2(\gamma - x)$ in (2.8) diverges as $h \to \frac{1}{2}$.

3. For $0 < h < \frac{1}{2}$ the leading term is $\frac{e^{-2h\phi}}{|\gamma - x|^{4h}}$. However, the integral $\int d^2\gamma\Phi_h$ over $|\gamma - x| < \epsilon$ does not vanish as $\epsilon \to 0$. Therefore, one cannot neglect the contribution $\frac{1}{2h-1}e^{2(h-1)\phi}\delta^2(\gamma - x)$ to Φ_h.

4. For $h = 0$ we have a more dramatic "resonance" and $\Phi_0 = 1$.

5. For $h < 0$ the leading term at large ϕ and generic γ, $\frac{e^{-2h\phi}}{|\gamma - x|^{4h}}$, diverges as $\phi \to \infty$.

6. For all values of h discussed above, $\Phi_h e^{\phi}$ grows as $\phi \to \infty$. Thus Φ_h is a non-normalizable operator, as is familiar from Liouville theory and the AdS/CFT correspondence.

Since (2.4) $J_0^- = -\frac{\partial}{\partial x}$, $\bar{J}_0^- = -\frac{\partial}{\partial \bar{x}}$, every observable $\Theta(x, \bar{x})$ is conjugate to $\Theta(0)$

$$\Theta(x, \bar{x}) = e^{-xJ_0^- - \bar{x}\bar{J}_0^-}\Theta(0)e^{xJ_0^- + \bar{x}\bar{J}_0^-} . \tag{2.9}$$

[5]For $\gamma = x$ the leading term diverges as $e^{2(h-1)\phi}\delta^2(0)$. This reflects the fact that the dominant term at $\gamma = x$ is $e^{2h\phi}$. However, since our vertex operators will always be integrated over, we should use the expression $e^{2(h-1)\phi}\delta^2(\gamma - x)$ and drop the term $e^{2h\phi}$.

Therefore it is natural to define

$$
\begin{aligned}
J^+(x;z) &= e^{-xJ_0^-} J^+(z) e^{xJ_0^-} = J^+(z) - 2xJ^3(z) + x^2 J^-(z) \\
J^3(x;z) &= e^{-xJ_0^-} J^3(z) e^{xJ_0^-} = J^3(z) - xJ^-(z) = -\frac{1}{2}\partial_x J^+(x;z) \\
J^-(x;z) &= e^{-xJ_0^-} J^-(z) e^{xJ_0^-} = J^-(z) = \frac{1}{2}\partial_x^2 J^+(x;z)\,.
\end{aligned} \tag{2.10}
$$

Since all these currents are related to each other by differentiation, it is enough to consider

$$
J(x;z) \equiv -J^+(x;z) = 2xJ^3(z) - J^+(z) - x^2 J^-(z)\,. \tag{2.11}
$$

An explicit computation shows that

$$
J(x;z) = k\left[(x-\gamma)^2 e^{2\phi}\partial_z\bar\gamma + 2(x-\gamma)\partial_z\phi - \partial_z\gamma\right]. \tag{2.12}
$$

Similar expressions can be written for $\bar J(\bar x;\bar z)$.

$J(x;z)$ is a spacetime tensor of weight $(-1,0)$ which, just like Φ_h, is non-singular for all x. The transformation properties (2.5) of Φ_h under $\mathrm{SL}(2)\times\overline{\mathrm{SL}(2)}$ can be summarized by the following action of the charges $J_0(x)$ associated with the conserved currents (2.11)

$$
[J_0(x), \Phi_h(y,\bar y)] = \left[(y-x)^2\partial_y + 2h(y-x)\right]\Phi_h(y,\bar y) \tag{2.13}
$$

and a similar relation for $\bar J_0$.

An observable which will be important below is

$$
\bar J(\bar x;\bar z)\Phi_1(x,\bar x;z,\bar z)\,. \tag{2.14}
$$

It is easy to check that

$$
\frac{\pi}{k}\bar J\Phi_1 = \partial_{\bar z}\Lambda\,, \tag{2.15}
$$

where

$$
\Lambda = -\frac{1}{\gamma - x}\frac{(\gamma-x)(\bar\gamma-\bar x)e^{2\phi}}{(\gamma-x)(\bar\gamma-\bar x)e^{2\phi}+1}\,. \tag{2.16}
$$

Since Φ_1 is a tensor of weight $(1,1)$ in spacetime, while $\bar J$ is a tensor of weight $(0,-1)$, (2.15) suggests that Λ is a tensor of weight $(1,0)$. However, the operators (2.7) are all left-right symmetric and as indicated above there are no solutions of (2.5) for $h\neq\bar h$. What is then the status of Λ of (2.16)? One can check that Λ transforms as follows under $\mathrm{SL}(2)\times\overline{\mathrm{SL}(2)}$

$$
\begin{aligned}
[J_0(x), \Lambda(y,\bar y)] &= \left[(y-x)^2\partial_y + 2(y-x)\right]\Lambda(y,\bar y) - 1 \\
\left[\bar J_0(\bar x), \Lambda(y,\bar y)\right] &= (\bar y-\bar x)^2\partial_{\bar y}\Lambda(y,\bar y).
\end{aligned} \tag{2.17}
$$

Comparing to (2.13) we see that Λ indeed transforms like an object with $\bar h = 0$ under $\overline{\mathrm{SL}(2)}$, while its transformation as an object with $h=1$ under SL(2) contains an anomalous term, which is a constant (independent of the worldsheet fields and of x, y). Thus, it is natural to expect that Λ is *not* a good observable in CFT on AdS_3. Later we will provide two independent arguments that support this conclusion. However, we will also see that the object of interest to us will be $\partial_{\bar{z}}\Lambda$ which, as is clear from (2.15), *is* a good observable in the theory.[6] This is consistent with the fact that the anomalous contribution to $[J_0^+, \Lambda]$, being constant, disappears when we differentiate with respect to $\bar{z}$ as in (2.15). We also note for later use that:

1. As we approach the boundary of AdS_3,
$$\lim_{\phi\to\infty} \Lambda = \frac{1}{x-\gamma}\,. \tag{2.18}$$

2. As functions on AdS_3, Λ and Φ_1 (2.7) are related by
$$\partial_{\bar{x}}\Lambda = \pi\Phi_1\,. \tag{2.19}$$

2.2 Quantum CFT on AdS_3

In the quantum theory, the currents (2.3) are generators of an $\widehat{\mathrm{SL}(2)} \times \overline{\widehat{\mathrm{SL}(2)}}$ algebra. Their OPE's are[7]

$$\begin{aligned}
J^3(z)J^\pm(w) &\sim \frac{\pm J^\pm(w)}{z-w} \\
J^3(z)J^3(w) &\sim -\frac{\frac{k}{2}}{(z-w)^2} \\
J^-(z)J^+(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}.
\end{aligned} \tag{2.20}$$

The operators Φ_h are primaries of the full $\widehat{\mathrm{SL}(2)} \times \overline{\widehat{\mathrm{SL}(2)}}$ algebra.[8] They satisfy

$$\begin{aligned}
J^3(z)\Phi_h(x,\bar{x};w,\bar{w}) &\sim -\frac{(x\partial_x + h)\Phi_h(x,\bar{x})}{z-w} \\
J^+(z)\Phi_h(x,\bar{x};w,\bar{w}) &\sim -\frac{(x^2\partial_x + 2hx)\,\Phi_h(x,\bar{x})}{z-w} \\
J^-(z)\Phi_h(x,\bar{x};w,\bar{w}) &\sim -\frac{\partial_x\Phi_h(x,\bar{x})}{z-w}
\end{aligned} \tag{2.21}$$

and similar OPE's with the $\bar{J}^A$. All operators in CFT on AdS_3 can be obtained by acting on the primaries Φ_h by the currents $J(x;z)$, (2.11), and $\bar{J}(\bar{x};\bar{z})$.

[6] As a good analogy consider the operators X and $\partial_{\bar{z}}X$ in two dimensional massless scalar field theory. While X is not a good observable, e.g. because its two point function is logarithmic in the separation, $\langle X(z)X(w)\rangle \sim \log|z-w|^2$, $\partial_{\bar{z}}X$ is a good observable.

[7] Here and below we denote by $\sim$ equality up to non-singular terms in an OPE.

[8] The operator that is usually referred to as a primary of $\widehat{\mathrm{SL}(2)}$ is $\Phi_h(x=0)$. $\Phi_h(x)$ is related to it by conjugation (2.9).

The stress tensor of the worldsheet CFT is given by the Sugawara form

$$T^{\rm ws}(z) = \frac{1}{k-2}\eta_{AB}J^A J^B = \frac{1}{k-2}\left[-(J^3)^2 + J^+J^-\right]. \tag{2.22}$$

$T^{\rm ws}$ satisfies the Virasoro algebra with $c = 3k/(k-2)$. There is a similar expression for $\overline{T}^{\rm ws}$ in terms of $\bar{J}$. The worldsheet scaling dimensions of Φ_h, $(\Delta_h, \bar{\Delta}_h)$ can be computed using (2.22). They are

$$\Delta_h = \bar{\Delta}_h = -\frac{h(h-1)}{k-2}. \tag{2.23}$$

The operator product expansions (2.20) and (2.21) can be expressed in a more compact form using the operator J of (2.11)

$$J(x;z)J(y;w) \sim k\frac{(y-x)^2}{(z-w)^2} + \frac{1}{z-w}\left[(y-x)^2\partial_y - 2(y-x)\right] J(y;w) \tag{2.24}$$

$$J(x;z)\Phi_h(y,\bar{y};w,\bar{w}) \sim \frac{1}{z-w}\left[(y-x)^2\partial_y + 2h(y-x)\right]\Phi_h(y,\bar{y}). \tag{2.25}$$

These expressions are in accord with the interpretation of $J(x)$ as J^+ conjugated to the point x (2.10). This point of view also makes it obvious why the OPE's (2.24) and (2.25) are regular when $x = y$.

The stress tensor (2.22) can be written directly in terms of the current $J(x;z)$ as

$$T^{\rm ws} = \frac{1}{2(k-2)}\left[J\partial_x^2 J - \frac{1}{2}(\partial_x J)^2\right]. \tag{2.26}$$

A few comments:

1. Using $\partial_x^3 J = 0$, $\partial_x T^{\rm ws} = 0$. The combination (2.26) corresponds to a tensor in spacetime of weight $(0,0)$.

2. The composite operator $J(x)\Phi_h(x,\bar{x})$ does not require normal ordering, and the classical manipulations in (2.14) and (2.15) are justified in the quantum theory. Thus (2.15) holds in the quantum theory (without any finite renormalizations). A related fact is that $J(x)\Phi_h(x,\bar{x})$ is a primary of worldsheet Virasoro (2.22).

3. Useful special cases of (2.24), (2.25) are

$$\begin{aligned} J(x;z)\Phi_1(y,\bar{y};w,\bar{w}) &\sim 6\frac{1}{z-w}\partial_y\left[(x-y)^2\Phi_1(y,\bar{y};w,\bar{w})\right] \\ J(x;z)\left[J(y;w)\Phi_1(y,\bar{y};w,\bar{w})\right] &\sim k(x-y)^2\partial_w\left[\frac{\Phi_1(y,\bar{y};w,\bar{w})}{z-w}\right]. \end{aligned} \tag{2.27}$$

4. In the free field Wakimoto representation [26], (2.11) takes the form ($\alpha_+ = \sqrt{2k-4}$)

$$J(x;z) = -\beta(x-\gamma)^2 + \alpha_+\partial_z\phi(x-\gamma) - k\partial_z\gamma\,, \tag{2.28}$$

which is related to the classical expression (2.12) by the appropriate rescaling of the variables, certain finite renormalizations, and the equations of motion of the Wakimoto representation, which imply $\beta = -ke^{\frac{2\phi}{\alpha_+}}\partial_z\bar{\gamma}$.

The two point function of Φ_h is determined by worldsheet conformal invariance and the $\mathrm{SL}(2) \times \overline{\mathrm{SL}(2)}$ action (2.21). It is

$$\langle \Phi_h(x,\bar{x};z,\bar{z})\Phi_{h'}(y,\bar{y};w,\bar{w})\rangle = \delta(h-h')\frac{D(h;k)}{|x-y|^{4h}|z-w|^{4\Delta_h}}. \tag{2.29}$$

The numerical coefficient $D(h;k)$ has been computed in [23]. Note that for $h = h'$ the two point function is infinite – it is proportional to $\delta(0)$. This infinite factor can be interpreted as the volume of the subgroup of the $\mathrm{SL}(2,\mathbb{C})$ isometry of our target space, which preserves x and y.

When considering string theory on AdS_3 one has to integrate over the locations of all vertex operators on the worldsheet and to divide by the volume of a subgroup of the two dimensional diffeomorphism group isomorphic to the Mobius group $\mathrm{SL}(2,\mathbb{C})_M$. For the zero point function, the CFT correlation function is proportional to the volume of the target space; i.e. to the volume of $\mathrm{SL}(2,\mathbb{C})$. Dividing by the volume of $\mathrm{SL}(2,\mathbb{C})_M$ we find a finite answer. Similarly, a one point function, if it does not vanish, is proportional to the subgroup of $\mathrm{SL}(2,\mathbb{C})$ which leaves one point in the target space invariant. Dividing by the volume of the subgroup of $\mathrm{SL}(2,\mathbb{C})_M$ which leaves a point on the worldsheet invariant we again find a finite answer. Finally, for the two point function, the CFT correlation function is proportional to $\delta(0)$, as above, but in string theory this infinity is cancelled by a similar factor from $\mathrm{SL}(2,\mathbb{C})_M$. All this is exactly as in non-critical string theory [25].

One can also compute the OPE of J with Λ by integrating (2.27) with respect to $\bar{y}$. The $\bar{y}$ independent integration constant can be determined by comparing with the semiclassical analysis (2.17), or by using the free field representation (2.28). This leads to

$$J(x;z)\Lambda(y,\bar{y};w,\bar{w}) \sim \frac{\left[(y-x)^2\partial_y + 2(y-x)\right]\Lambda(y,\bar{y};w,\bar{w}) - 1}{z-w}. \tag{2.30}$$

We can now provide an independent argument that Λ itself is not a good observable in the theory. According to (2.15), (2.19), (2.29) the two point function of Λ satisfies

$$\langle \Lambda(x;z)\Lambda(y;w)\rangle = \frac{a\log|z-w|^2 + b\log|x-y|^2}{(x-y)^2}, \tag{2.31}$$

where a and b are infinite constants that become finite when passing to string theory, as discussed above. We have also used the fact that correlation functions are single valued both on the worldsheet and in spacetime. The correlator (2.31) depends logarithmically on the separation, both on the worldsheet and in spacetime. Hence, like X in free field theory, Λ is not a good observable.

Another argument that Λ cannot be a good observable in CFT on AdS_3 follows from the relation (2.19). Φ_1 is used in string theory on AdS_3 to dress marginal operators in the spacetime CFT. If Λ was a good observable, all dimension $(1,1)$ operators in the spacetime CFT would have been total derivatives with respect to $\bar{x}$ of operators with scaling dimensions $(1,0)$. This is highly unlikely based on the available information about particular examples of string theory on AdS_3 (see e.g. [4] for details). If one further assumes that the spacetime CFT is unitary, any operator of dimension $(1,0)$ must be holomorphic (in x), and therefore we would have concluded that there are *no* marginal operators at all in the spacetime CFT, which again contradicts available information.

We next turn to the OPE algebra of the $\widehat{\mathrm{SL}(2)}$ primaries Φ_h. De Boer et. al. [5] studied the operator product $\Phi_{h_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{h_2}(x_2,\bar{x}_2;z_2,\bar{z}_2)$ in the semiclassical region of large k, for h_2 of order k and h_1 of order one. The insertion of Φ_{h_2} affects the worldsheet and takes $\phi(z_2)\to\infty$ and $\gamma(z_2)\to x_2$. Therefore, the operator product expansion is dominated by large ϕ and $\gamma\approx x_2$,

$$\Phi_{h_1}(x_1;z_1)\Phi_{h_2}(x_2;z_2)\sim |z_1-z_2|^{-4\frac{(h_1-1)(h_2-1)}{k-2}}\delta^2(x_1-x_2)\Phi_{h_1+h_2-1}(x_2;z_2)\,, \tag{2.32}$$

where we inserted the factor $|z_1-z_2|^{-4\frac{(h_1-1)(h_2-1)}{k-2}}$ on dimensional grounds. We can also study this operator product when $h_1,h_2\ll k$ in the semiclassical region. In this limit the dimensions of all Φ_h are much smaller than one. Therefore, the leading contribution comes from terms with no derivatives in the operator product expansion $\Phi_{h_1}\Phi_{h_2}$. Terms with derivatives have larger dimensions. At large ϕ, the product behaves like

$$\begin{aligned}\Phi_{h_1}\Phi_{h_2} =&\ e^{2(h_1+h_2-2)\phi}\delta^2(\gamma-x_1)\delta^2(\gamma-x_2)+\mathcal{O}(e^{2(h_1+h_2-3)\phi})\\ =&\ e^{2(h_1+h_2-2)\phi}\delta^2(x_1-x_2)\delta^2(\gamma-x_2)+\mathcal{O}(e^{2(h_1+h_2-3)\phi}).\end{aligned} \tag{2.33}$$

Therefore, it is approximately $\Phi_{h_1}(x_1;z_2)\Phi_{h_2}(x_2;z_2)=\delta^2(x_1-x_2)\Phi_{h_1+h_2-1}+\cdots$ where the ellipses stand for operators with $\Re h<h_1+h_2-1$. When k is large but not infinite the dimensions of the operators do not vanish and the right hand side should be multiplied by $|z_1-z_2|^{-4\frac{(h_1-1)(h_2-1)}{k-2}}$, as in (2.32). The operators with $\Re h<h_1+h_2-1$ have larger dimension than $\Phi_{h_1+h_2-1}$ and therefore they are negligible as $z_1\to z_2$. We conclude that at least for $h_1\ll k$ and $k\gg 1$

$$\begin{aligned}\Phi_{h_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{h_2}(x_2,\bar{x}_2;z_2,\bar{z}_2)=C_{12}|z_1-z_2|^{-4\frac{(h_1-1)(h_2-1)}{k-2}}\delta^2(x_1-x_2)\times\\ \times\Phi_{h_1+h_2-1}(x_2,\bar{x}_2;z_2,\bar{z}_2)+\text{smaller terms as } z_1\to z_2.\end{aligned} \tag{2.34}$$

For small values of h the asymptotic form of the vertex operators is different (see the discussion above) and this conclusion could be modified. A special case of (2.35) is

$$\lim_{z_1\to z_2}\Phi_1(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_h(x_2,\bar{x}_2;z_2,\bar{z}_2)=\delta^2(x_1-x_2)\Phi_h(x_2,\bar{x}_2;z_2,\bar{z}_2)\,. \tag{2.35}$$

An important aspect of (2.35) is that the coefficient on the right hand side is independent of h and therefore we can normalize Φ_1, as in (2.8), such that it is equal to one.

In the exact theory the situation is more complicated. The OPE structure is [23]

$$\begin{aligned}\Phi_{h_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{h_2}(x_2,\bar{x}_2;z_2,\bar{z}_2) &= \\ &= \int dh_3 \int d^2x_3 \frac{f_{h_1h_2h_3}(x_i,\bar{x}_i)\Phi_{h_3}(x_3,\bar{x}_3;z_2,\bar{z}_2)}{|z_1-z_2|^{2(\Delta_1+\Delta_2-\Delta_3)}} + \text{descendants}\,.\end{aligned} \tag{2.36}$$

The OPE coefficients $f_{h_1h_2h_3}(x_i,\bar{x}_i)$ can be computed by studying the three point functions of Φ_{h_j} in CFT on AdS_3. They have two components. The one that is studied in [23] corresponds to f_{123} which is a smooth function of x_i and of h_i. Later we will be interested in computing the OPE (2.35) in the exact theory. Smooth f's do not contribute to this quantity since one could continue analytically in h from a region where the limit of (2.35) as $z_1 \to z_2$ vanishes.

A second class of f's corresponds to distributions in x_i. In particular, we can examine the possibility that

$$\Phi_{h_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{h_2}(x_2,\bar{x}_2;z_2,\bar{z}_2) \sim \frac{f(x_1,x_2)}{|z_1-z_2|^{2(\Delta_1+\Delta_2-\Delta_3)}}\Phi_{h_3}(x_2,\bar{x}_2;z_2,\bar{z}_2)\,. \tag{2.37}$$

In that case, SL(2) invariance constrains severely the function $f(x_1,x_2)$ and (2.35) is one of the only consistent solutions of the SL(2) Ward identity. Indeed, applying $\oint J(y)$ to the two sides of the OPE (2.37) where the contour surrounds z_1, z_2, we find that f must satisfy

$$\Big[(x_1-y)^2\partial_{x_1} + (x_2-y)^2\partial_{x_2} + 2h_1(x_1-y) + 2h_2(x_2-y) - 2h_3(x_2-y)\Big]\, f = 0\,. \tag{2.38}$$

The coefficient of y^2 in (2.38) (the J^- Ward identity) shows that f depends only on $x_1 - x_2$. The coefficient of y in (2.38) (the J^3 Ward identity) leads to

$$[(x_1-x_2)\partial_{x_1} + (h_1+h_2-h_3)]f = 0\,. \tag{2.39}$$

Therefore, if f is not a distribution,

$$f = (x_1-x_2)^{h_3-h_1-h_2}\,. \tag{2.40}$$

The term without y in (2.38) is then satisfied only if $h_2 = h_1 + h_3$. For the particular case of interest in (2.35), $h_1 = 1$, we find that $h_2 = h_3 + 1$, and the power of $|z_1 - z_2|$ in (2.37) is $4h_3/(k-2) > 0$. Thus, such an f does not contribute to the limit (2.35).

If we also allow f which is a distribution, (2.39) can be satisfied by

$$f = \partial_{x_1}^n \partial_{\bar{x}_1}^{\bar{n}} \delta^2(x_1-x_2)\,, \qquad \text{for } h_1+h_2-h_3 = n+1\,. \tag{2.41}$$

A similar relation with $\bar{J}$ sets $n = \bar{n}$. The term with no y in (2.38) further allows the answer to be of this form only for $n = 0$, $h_1 + h_2 - h_3 = 1$ or $h_1 = (n+1)/2$, $h_3 = h_2 - (n+1)/2$.

To summarize, we see that terms of the form (2.37) in the OPE $\Phi_1\Phi_h$ give a finite contribution in the limit $z_1 \to z_2$ only if x_1 is in the vicinity of x_2. The corresponding OPE coefficient is proportional to $\delta^2(x_1 - x_2)$, as in (2.35). Note that it is not guaranteed a-priori that one can choose the numerical prefactor on the right hand side of (2.35) to be one (it could depend on h). We saw that this prefactor can be chosen to be one in the semiclassical limit, and we will assume that this continues to hold in the exact theory. This assumption is necessary for the spacetime Ward identities.

The above discussion also shows that the divergence of the OPE coefficient in (2.35), $\delta^2(x_1 - x_2)$, as $x_1 \to x_2$ is directly related to the non-compactness of AdS_3. Short distance singularities in the spacetime CFT come from $\phi \to \infty$ on AdS_3. As discussed in the introduction, this is an example of the UV/IR correspondence of [20].

Another description of CFT on AdS_3 is obtained by "Fourier transforming" the local fields Φ_h and defining the mode operators

$$V_{j;m,\bar{m}} = \int d^2x x^{j+m} \bar{x}^{j+\bar{m}} \Phi_{j+1}(x,\bar{x})\,. \tag{2.42}$$

The inverse transformation is

$$\Phi_h(x,\bar{x}) = \sum_{m,\bar{m}} V_{h-1;m,\bar{m}} x^{-m-h} \bar{x}^{-\bar{m}-h}\,. \tag{2.43}$$

Equations (2.42), (2.43) are essentially mode expansions of the local spacetime fields in the dual CFT and, just as in standard CFT, the local fields are the fundamental objects and the modes may or may not be well defined and useful. Indeed, in (2.43) it is in general not clear what range the variables $(m, \bar{m})$ should run over.[9] In (2.42) the integral over x is in general divergent and is defined by finding a region where it converges and analytically continuing from there. This may or may not be possible in different cases. Thus, we will here work with local fields in x space and not with their modes. The modes $V_{j;m,\bar{m}}$ have the scaling dimensions (2.23) with $h = j + 1$. They satisfy the OPE algebra with the currents (which is equivalent to (2.21), (2.25)

$$\begin{aligned} J^3(z)V_{j;m,\bar{m}}(w,\bar{w}) &\sim \frac{mV_{j;m,\bar{m}}}{z-w} \\ J^\pm(z)V_{j;m,\bar{m}}(w\bar{w}) &\sim \frac{(m \mp j)V_{j;m\pm 1,\bar{m}}}{z-w} \end{aligned} \tag{2.44}$$

and similarly for $\bar{J}^A(\bar{z})$. In addition to primaries, the theory contains descendants which have higher order poles in their OPE with J^A. The latter can be described as (normal ordered) products of currents and their derivatives with $V_{j;m,\bar{m}}$.

[9] Locality in $(x,\bar{x})$ space implies that $m - \bar{m} \in \mathbb{Z}$, but $m + \bar{m}$ is not constrained.

3. The spacetime current algebra

We next turn to (bosonic) string theory on $AdS_3 \times \mathcal{N}$, with $\mathcal{N}$ an arbitrary compact manifold which together with AdS_3 provides a solution of the string equations of motion (see [4] for a more detailed discussion). Assume that the CFT on $\mathcal{N}$ contains a worldsheet current algebra associated with some group G, generated by worldsheet currents $k^a(z)$ satisfying

$$k^a(z)k^b(w) \sim \frac{\frac{1}{2}k_G\delta^{ab}}{(z-w)^2} + \frac{f^{abc}k^c(w)}{z-w}\,, \tag{3.1}$$

where k_G is the level of the worldsheet affine Lie algebra $\widehat{G}$ and $\frac{1}{i}f^{abc}$ are the (real) structure constants of G. One can then construct in the spacetime theory currents $K^a(x)$, which satisfy an algebra similar to (3.1). The purpose of this section is to construct the generators of this spacetime current algebra, following [4]. On general grounds one expects the local field $K^a(x)$ to be described by the vertex operator of an almost pure gauge field. We start with a review of some relevant properties of gauge fields in string theory.

Associated with each worldsheet current $k^a(z)$ satisfying (3.1) one finds in string theory a target space gauge field whose vertex operator is

$$\int d^2z A^a_\mu(x)k^a(z)\partial_{\bar{z}}x^\mu\,, \tag{3.2}$$

where x^μ are coordinates on the target space.[10] Under infinitesimal gauge transformations the gauge field A^a_μ transforms as[11]

$$\delta A^a_\mu = \partial_\mu\lambda^a\,. \tag{3.3}$$

Thus, a pure gauge field (3.2) is described by the vertex operator

$$\int d^2z k^a(z)\partial_{\bar{z}}\lambda^a(x^\mu)\,, \tag{3.4}$$

where λ^a is the gauge function. The fact that the theory is gauge invariant implies that couplings like (3.4) do not influence the physics, however this is only true for gauge functions λ^a with compact support in target space. As we will see in detail below, gauge transformations with gauge function λ^a that do not vanish at the boundary of AdS_3 generate a large algebra of global transformations, the affine Lie algebra $\widehat{G}$.

There is an important point that should be mentioned here. Eqs. (3.2), (3.4) describe gauge field couplings in the σ-model. In string theory, one has to impose

[10] In our case the target space is AdS_3, and $(x^\mu) = (\phi, \gamma, \bar{\gamma})$.

[11] For simplicity, we neglect the non-linear term in the gauge transformation, i.e. assume that A is small.

the worldsheet consistency conditions. In particular, the integrands of (3.2), (3.4) must be primaries under the left and right-moving worldsheet Virasoro symmetries with scaling dimension $(1,1)$. When the target space is flat, this implies that the gauge field satisfies the gauge condition $\partial^\mu A^a_\mu = 0$ (as well as the massless Klein-Gordon equation $\partial_\nu \partial^\nu A^a_\mu = 0$). The remaining gauge freedom corresponds to gauge functions satisfying $\partial_\mu \partial^\mu \lambda^a = 0$. If the gauge function λ^a has compact support, it can be Fourier transformed. The corresponding operators are $\int d^2 z k^a(z) \partial_{\bar z} e^{ik\cdot x}$ with $k^2 = 0$. Moding out by BRST commutators removes such operators.

Another set of "pure gauge" operators satisfying $\partial_\mu \partial^\mu \lambda^a = 0$ is associated with $\lambda^a = x^\mu$ and $\lambda^a = 1$. These gauge functions do not go to zero at infinity. The corresponding vertex operators, $\int d^2 z k^a(z) \partial_{\bar z} x^\mu$ and $\oint dz k^a(z)$, are in the BRST cohomology. The status of the above two kinds of operators in string theory is actually somewhat different. The gauge functions $\lambda^a = x^\mu$ are not good operators in the worldsheet conformal field theory. This can be seen by noticing that their two point functions depend on the logarithm of the separation. Therefore, the corresponding "pure gauge" vertex operators are not BRST exact and give rise to physical vertex operators (the zero momentum gluons). The gauge function $\lambda^a = 1$ does correspond to a good operator on the worldsheet (the identity operator), and the corresponding vertex operator $\int d^2 z k^a(z) \partial_{\bar z} 1$ vanishes. Its vanishing is the statement that all correlation functions in string theory are invariant under the global part of the gauge symmetry.

Returning to string theory on AdS_3, we expect the spacetime affine currents to have the form

$$K^a(x) = \int d^2 z k^a(z) \partial_{\bar z} \lambda(\gamma, \bar\gamma, \phi; x, \bar x)\,, \tag{3.5}$$

with a gauge function λ that does not go to zero as $\phi \to \infty$. Since $K^a(x)$ has spacetime scaling dimension $h = 1$, $\bar h = 0$, we are looking for an operator $\partial_{\bar z}\lambda$ with worldsheet scaling dimension $(\Delta, \bar\Delta) = (0,1)$ and spacetime scaling dimension $(h, \bar h) = (1,0)$. We constructed precisely such an operator in (2.15), (2.16). Below we will determine the normalization $\lambda = -\Lambda/\pi$,

$$K^a(x) = -\frac{1}{\pi} \int d^2 z k^a(z) \partial_{\bar z} \Lambda(\gamma, \bar\gamma, \phi; x, \bar x)\,. \tag{3.6}$$

The spacetime current (3.6) is formally pure gauge, but the gauge function Λ does not have compact support, and is not a good observable in CFT on AdS_3. The corresponding vertex operator $K^a(x)$ does not decouple from correlation functions. Another useful way of writing $K^a(x)$ (using (2.15)) is

$$K^a(x) = -\frac{1}{k} \int d^2 z k^a(z) \bar J(\bar x; \bar z) \Phi_1(x, \bar x; z, \bar z)\,, \tag{3.7}$$

which makes manifest that $K^a(x)$ is physical and has weight $(1,0)$ in spacetime.

From the point of view of the spacetime theory we expect the current $K^a(x)$ (3.6) to be holomorphic, up to contact terms. Differentiating (3.6) with respect to $\bar{x}$ and using (2.19) gives

$$\partial_{\bar{x}} K^a = -\int d^2 z k^a(z) \partial_{\bar{z}} \Phi_1 \,. \tag{3.8}$$

Unlike (3.6), $\partial_{\bar{x}} K^a$ given by (3.8) is a total derivative of a good observable in CFT on AdS_3, and we expect it to decouple (for generic x). Indeed we will see in the next section that (3.8) vanishes inside correlation functions, except for contact term contributions at the locations of other operators in the correlation function. These contact terms give rise to the affine Lie algebra Ward identities familiar from two dimensional CFT.

To summarize, "pure gauge" vertex operators (3.4) in string theory belong to one of a few classes, depending on properties of the gauge function $\lambda(x)$. If λ is normalizable, (3.4) is null and decouples. This is the situation usually considered in string theory. If λ is non-normalizable but is a good observable in the worldsheet CFT, like Φ_1 (3.8), the pure gauge field (3.4) decouples "generically," e.g. for generic momenta or for generic x in the AdS_3 example (3.8). If the gauge function λ is not a good observable in the worldsheet theory, but $\partial_{\bar{z}}\lambda$ is (as in (3.6), (3.7)), the gauge field (3.4) does not decouple and is in the BRST cohomology. Gauge functions λ for which $\partial_{\bar{z}}\lambda$ is not a good observable (e.g. $\lambda = x^\mu x^\nu$ in flat spacetime) should not be considered, since in that case (3.4) is not a physical operator.

4. The current algebra Ward identity

Now that we have constructed the spacetime affine Lie algebra generator $K^a(x)$ we would like to study its correlation functions. The typical correlation function of interest is

$$\langle K^a(x) K^b(y) W_h(s,\bar{s}) \cdots \rangle \,, \tag{4.1}$$

where $W_h(s,\bar{s};w,\bar{w})$ is an operator that transforms in some representation R of the worldsheet current algebra,

$$k^a(z) W_h(s,\bar{s};w,\bar{w}) \sim \frac{t^a(R) W_h(s,\bar{s};w,\bar{w})}{z-w} \,. \tag{4.2}$$

$t^a(R)$ are the representation matrices of the Lie algebra of G in the representation R. h labels the transformation properties of W_h under the worldsheet $\widehat{\mathrm{SL}(2)} \times \overline{\widehat{\mathrm{SL}(2)}}$ algebra; in other words,[12] $W_h(s)$ includes a factor of $\Phi_h(s)$. General considerations

[12]For simplicity, we will restrict to W_h's that are primary under worldsheet $\widehat{\mathrm{SL}(2)} \times \overline{\widehat{\mathrm{SL}(2)}}$. As explained in [4], there are many physical operators that are descendants of worldsheet current algebra but are physical in string theory (i.e. primary under worldsheet Virasoro).

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in the spacetime CFT imply that the correlation function (4.1) should satisfy the standard current algebra Ward identities,

$$\begin{aligned}
K^a(x)K^b(y) &\sim \frac{\frac{1}{2}k_G^{(st)}\delta^{ab}}{(x-y)^2} + \frac{f^{abc}K^c(y)}{x-y} \\
K^a(x)W_h(s,\bar{s}) &\sim \frac{t^a(R)W_h(s,\bar{s})}{x-s}\,.
\end{aligned} \tag{4.3}$$

Our main task in this section is to derive this Ward identity in string theory on AdS_3, and to compute the level of the spacetime current algebra $k_G^{(st)}$.

The strategy of the calculation is the following. The integral in (3.6) runs over the whole worldsheet, Σ. We would like to write $K^a(x)$ as an integral of a total derivative. Since $\partial_{\bar{z}}k^a(z) = 0$ except for delta function contributions at the locations on the worldsheet of the other insertions in the correlator, we can write the integrand of (3.6) as a total derivative if we restrict the worldsheet integral to Σ', the Riemann surface Σ with small holes around the insertions removed. The integral (3.6) over such holes goes to zero as their size is taken to zero; therefore, extracting the holes does not change the value of correlators like (4.1) and amounts to a particular regularization.

After extracting the holes we can rewrite (3.6) as

$$K^a(x) = -\frac{1}{\pi}\int_{\Sigma'} d^2z\partial_{\bar{z}}\left[k^a(z)\Lambda(x,\bar{x};z,\bar{z})\right]. \tag{4.4}$$

If the worldsheet correlators of Λ are well behaved, we can perform the integral in (4.4) and it will receive contributions only from the boundaries of the Riemann surface with holes Σ'. This leads to the representation

$$K^a(x) = \sum_i \oint_{C_i} \frac{dz}{2\pi i} k^a(z)\Lambda(x,\bar{x};z,\bar{z})\,, \tag{4.5}$$

where C_i is the boundary of the small hole around the i'th insertion.

Previously we have argued that Λ is not a good observable in CFT on AdS_3; e.g. it has a logarithmically divergent two point function (2.31). However, the question that is relevant for analyzing the validity of (4.5) is how $\Lambda(z,\bar{z})$ behaves as one approaches an insertion involving Φ_h or some (worldsheet) current algebra descendant thereof. To see that near such insertions Λ is well behaved, one can integrate (2.35) with respect to $\bar{x}_1$, using

$$\partial_{\bar{x}}\frac{1}{x-y} = \pi\delta^2(x-y)\,, \tag{4.6}$$

and fix the $\bar{x}_1$ independent integration constant by a combination of semiclassical and free field techniques. This leads to

$$\lim_{z_1\to z_2}\Lambda(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_h(x_2,\bar{x}_2;z_2,\bar{z}_2) \sim \frac{1}{x_1-x_2}\Phi_h(x_2,\bar{x}_2;z_2,\bar{z}_2)\,. \tag{4.7}$$

Alternatively, one can differentiate (4.5) with respect to $\bar{x}$ and use (2.19). One finds

$$\partial_{\bar{x}} K^a(x) = \sum_i \oint_{C_i} \frac{dz}{2i} k^a(z) \Phi_1(x,\bar{x};z,\bar{z}) \,, \tag{4.8}$$

which is equivalent to (3.8). Here there are no subtleties regarding the existence of the right hand side in CFT on AdS_3. The two representations (4.5), (4.8) are of course equivalent but below we will find it convenient to use (4.5) in the asymptotic analysis near $\phi = \infty$ (in section 7) and (4.8) in the discussion of the exact theory (in this section).

Note that (4.8) and the related form (3.8) provide an example of the worldsheet–spacetime relation discussed in [4]. The left hand side of (3.8) is naively zero since $K^a(x)$ is a holomorphic current in spacetime. Correlation functions of $\partial_{\bar{x}} K^a$ such as (4.1) receive contact term contributions from the boundaries of spacetime, which correspond to x approaching other insertions. These contributions are governed by the spacetime Ward identities. The right hand side of (3.8) is naively zero since $k^a(z)$ is a holomorphic current.[13] Correlation functions of the right hand side of (3.8) receive contributions from the boundaries of the worldsheet, which correspond to z approaching other insertions. These contributions are governed by the worldsheet Ward identity.

In the rest of this section we will derive the spacetime Ward identity for removing $K^a(x)$ from correlation functions like (4.1) by using the representation (4.8) for K^a. Substituting $\partial_{\bar{x}} K^a(x)$ (4.8) in (4.1) we have to analyze the contributions to the correlator from the boundaries of the worldsheet, C_i. Consider the contribution from a small circle around the location of $W_h(s,\bar{s})$

$$\langle \partial_{\bar{x}} K^a(x) W_h(s) \cdots \rangle \sim \int d^2 w \oint_w \frac{dz}{2i} \Big\langle k^a(z) \Phi_1(x,\bar{x};z,\bar{z}) W_h(s,\bar{s};w,\bar{w}) \cdots \Big\rangle \,. \tag{4.9}$$

Using (2.35), (4.2) and performing the integral over z leads to

$$\langle \partial_{\bar{x}} K^a(x) W_h(s) \cdots \rangle \sim \pi \delta^2(x-s) t^a(R) \langle W_h(s) \cdots \rangle \,, \tag{4.10}$$

which is the correct result (4.3) (using (4.6)).

The contribution to the correlator (4.1) from a small circle surrounding the location of another current, $K^b(y)$, can be computed similarly

$$\begin{aligned} \langle \partial_{\bar{x}} K^a(x) K^b(y) \cdots \rangle \sim -\frac{1}{k} \int d^2 w \oint_w \frac{dz}{2i} \Big\langle k^a(z) \Phi_1(x,\bar{x};z,\bar{z}) k^b(w) \bar{\times} \\ \times J(\bar{y};\bar{w}) \Phi_1(y,\bar{y};w,\bar{w}) \cdots \Big\rangle \,. \end{aligned} \tag{4.11}$$

There are now two contributions, from the single and double pole in the worldsheet OPE (3.1). The single pole is treated in a similar way to the one discussed for W_h

[13] Or, equivalently, the right hand side of (3.8) is BRST exact on the worldsheet.

above. The only new element here is that there might be an additional contribution from the OPE of $\bar{J}(\bar{y};\bar{w})$ with $\Phi_1(x,\bar{x};z,\bar{z})$. Since that OPE has a positive power of $x-y$ and it multiplies a $\delta^2(x-y)$ (2.35), this contribution vanishes. Thus in direct analogy to (4.10) one finds that the single pole contribution gives

$$\langle \partial_{\bar{x}} K^a(x) K^b(y) \cdots \rangle = \pi \delta^2(x-y) f^{abc} \langle K^b(y) \cdots \rangle + \cdots . \tag{4.12}$$

The double pole contribution to (4.11) gives an operator proportional to

$$\lim_{z \to w} \partial_z \Phi_1(x,\bar{x};z,\bar{z}) \bar{J}(\bar{y};\bar{w}) \Phi_1(y,\bar{y};w,\bar{w}) , \tag{4.13}$$

which we would like to compute using (2.35). Unfortunately, the limit (4.13) is sensitive to the $O(z-w)$ term in (2.35) which of course is suppressed as $z \to w$. Nevertheless, we can make use of (2.35), by using the fact that

$$\partial_z \Phi_1(x,\bar{x};z,\bar{z}) = \frac{1}{k} \partial_x \left[J(x;z) \Phi_1(x,\bar{x};z,\bar{z}) \right] . \tag{4.14}$$

This can be shown by computing the OPE of the worldsheet stress tensor (2.22) with Φ_1 using (2.21) and isolating the single pole term, which by definition is $\partial_z \Phi_1$. An alternative proof of (4.14) is obtained by using the fact that $\pi \Phi_1 = \partial_x \bar{\Lambda}$ (the complex conjugate of (2.19)); thus

$$\partial_z \Phi_1 = \frac{1}{\pi} \partial_x \partial_z \bar{\Lambda} = \frac{1}{k} \partial_x (J \Phi_1) , \tag{4.15}$$

where in the last step we used the complex conjugate of (2.15). Subsituting this form of $\partial_z \Phi_1$ in (4.13) and using (2.35) we find that

$$\begin{aligned} k \lim_{z \to w} & \partial_z \Phi_1(x,\bar{x};z,\bar{z}) \bar{J}(\bar{y};\bar{w}) \Phi_1(y,\bar{y};w,\bar{w}) = \\ & = \lim_{z \to w} \partial_x \left[J(x;z) \Phi_1(x,\bar{x};z,\bar{z}) \bar{J}(\bar{y};\bar{w}) \Phi_1(y,\bar{y};w,\bar{w}) \right] \\ & = \partial_x \delta^2(x-y) J(y;w) \bar{J}(\bar{y};\bar{w}) \Phi_1(y,\bar{y};w,\bar{w}) . \end{aligned} \tag{4.16}$$

Finally, collecting all the terms and coefficients, we find that the spacetime currents satisfy

$$\langle \partial_{\bar{x}} K^a(x) K^b(y) \cdots \rangle \sim \pi \delta^2(x-y) f^{abc} \langle K^b(y) \cdots \rangle - \pi \frac{1}{2} k_G \delta^{ab} \partial_x \delta^2(x-y) \langle I \cdots \rangle , \tag{4.17}$$

where the operator I is given by

$$I = \frac{1}{k^2} \int d^2 z J(x;z) \bar{J}(\bar{x};\bar{z}) \Phi_1(x,\bar{x};z,\bar{z}) . \tag{4.18}$$

Thus, the currents $K^a(x)$ satisfy the spacetime OPE algebra

$$K^a(x) K^b(y) \sim \frac{f^{abc}}{x-y} K^c(y) + \frac{\frac{1}{2} I k_G \delta^{ab}}{(x-y)^2} . \tag{4.19}$$

In the next section we discuss the operator I (4.18) that appears in the spacetime current algebra, and study the consequences of (4.19) for correlation functions in string theory on AdS_3.

5. Properties of the central extension

The discussion of the previous sections makes it clear that the operator I (4.18) has the following properties:

1. It is physical (BRST invariant) in string theory on AdS_3. Naively I is BRST exact because of (2.159, but like the spacetime current $K^a(x)$ it does not decouple, for the same reasons.

2. As is clear from the form of the vertex operator, I is essentially a mode of the dilaton/graviton field.

3. It transforms as a tensor of weight $(0,0)$ in spacetime, i.e. its scaling dimensions in the spacetime CFT are $h = \bar{h} = 0$. Therefore, we intepret it as (a multiple of) the identity operator of the spacetime CFT.

For this interpretation to be consistent, I must be a constant, $\partial_x I = \partial_{\bar{x}} I = 0$. To prove that I is indeed independent of x, $\bar{x}$, one can use considerations similar to those of the previous sections. By using (2.15) we can write $\partial_{\bar{x}} I$ (up to a constant, which is irrelevant for the argument) as

$$\partial_{\bar{x}} I \propto \int d^2 z J(x;z) \partial_{\bar{z}} \Phi_1 \propto \int d^2 z \partial_{\bar{z}} \left[J(x;z) \Phi_1 \right] \propto \oint dz \partial_z \bar{\Lambda} \,. \tag{5.1}$$

So, $\partial_{\bar{x}} I$ is given by an integral of a total derivative over the boundary of the worldsheet. This will vanish, unless there is a (worldsheet) singularity in the OPE of the operator $\bar{\Lambda}$ with other physical operators in the theory. But we have used above the fact that such OPE's are in fact regular (see e.g. (4.7)). Hence, the operator (5.1) actually decouples in all correlation functions and we conclude that $\partial_{\bar{x}} I = 0$. A similar argument can be used to show that $\partial_x I = 0$.

The fact that I is a constant means that it is not a vertex operator of a scalar particle in AdS. Above we noticed that it is a mode of the spacetime dilaton field. These two assertions are consistent because the dilaton is a fixed scalar and hence it is massive. As a massive particle it does not have an on-shell zero mode. I is a physical on-shell zero mode of the dilaton field. It is related to the integration constant discussed in section 1. Below we will write the vertex operator of the massive dilaton.

This similarity and distinction between the massive dilaton and I is identical to the relation between the massive gauge bosons and the current algebra vertex operators $K^a(x)$. The latter are vertex operators of modes of the gauge fields, which should not be confused with the vertex operators of the massive gauge fields. Below we will write the vertex operators of the massive gauge fields.

Could the constant I be zero? The two point function $\langle I(x)I(y)\rangle$ is proportional to $\langle\Phi_1(x,\bar{x};z,\bar{z})\Phi_1(y,\bar{y};w,\bar{w})\rangle$ which cannot vanish.[14] Therefore, I is a non-zero multiple (which will be computed below) of the identity operator of the spacetime theory.

To see the consequences of the OPE (4.19) for correlation functions in string theory on AdS_3, consider a correlation function of the form

$$G^{ab}(x,y,x_1,\cdots,x_n)\equiv\langle\langle K^a(x)K^b(y)V_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle\,, \tag{5.2}$$

where $V_{h_j}(x_j)$ are any physical operators in string theory on AdS_3, and the double bracket notation implies that (5.2) receives contributions from all worldsheet topologies including disconnected ones (this point was emphasized in [5]). As $x\to y$, G^{ab} exhibits the following behavior

$$\begin{aligned}G^{ab}(x,y,x_1,\cdots,x_n) = &\frac{\frac{1}{2}k_G\delta^{ab}}{(x-y)^2}\langle\langle IV_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle + \\ &+\frac{f^{abc}}{x-y}\langle\langle K^c(y)V_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle+\text{finite}\,.\end{aligned} \tag{5.3}$$

Thus, the level of the spacetime current algebra $k_G^{(st)}$ is

$$k_G^{(st)}=k_G P(g_s)\,, \tag{5.4}$$

where $P(g_s)$ is defined by

$$\langle\langle IV_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle = P(g_s)\langle\langle V_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle\,. \tag{5.5}$$

This is the statement that I is a multiple of the identity operator. In order to explore (5.5) and to express $P(g_s)$ in terms of correlation functions of I, we consider for simplicity a case where the disconnected correlation functions of the V_{h_i} all vanish. It is simple to generalize the discussion to the case where they do not vanish. Then,

$$\begin{aligned}\langle\langle IV_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle = &\langle\langle I\rangle\rangle\langle\langle V_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle + \\ &+\langle\langle IV_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle_{\text{connected}}\,,\end{aligned} \tag{5.6}$$

where the last term is a sum over all genera but only of connected worldsheets. Comparing (5.5) an (5.6) we see that connected correlation functions must satisfy

$$\langle\langle IV_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle_{\text{connected}} = I(g_s)\langle\langle V_{h_1}(x_1)\cdots V_{h_n}(x_n)\rangle\rangle_{\text{connected}}\,, \tag{5.7}$$

with $I(g_s)$ independent of the operators V_{h_i}. This is a strong assumption, which we

[14] Vertex operators of marginal operators in the spacetime CFT are products of marginal operators on the worldsheet and Φ_1. Their two point functions lead to the Zamolodchikov metric and are proportional to $\langle\Phi_1\Phi_1\rangle$. Thus the latter cannot vanish.

do not know how to prove, but below we will give some evidence that it is true. Equations (5.5), (5.6) and (5.7) imply that

$$P(g_s) = \langle\langle I \rangle\rangle + I(g_s)\,. \tag{5.8}$$

To leading order in g_s (or equivalently $1/p$ (1.5)), $P(g_s) = \langle I \rangle_0 = I_0 Z_0$, where I_0 is the normalized one point function of I on the sphere,

$$I_0 \equiv \frac{\langle I \rangle_0}{Z_0}\,, \tag{5.9}$$

where Z_0 is the spherical partition sum. Similarly, the leading order form of (5.7) is

$$\langle I V_{h_1}(x_1) \cdots V_{h_n}(x_n) \rangle_0 = I_0 \langle V_{h_1}(x_1) \cdots V_{h_n}(x_n) \rangle_0\,, \tag{5.10}$$

from which we see that the leading behavior of $I(g_s)$ as $g_s \to 0$ is $I(g_s) = I_0 + O(g_s^2)$. The leading term is fixed by setting all V_h to one in (5.10) and comparing to (5.9).

The assumption (5.7) can be tested and the value of I_0 can be determined by comparing different correlation functions. Consider for example the following three point functions[15] and use (5.10)

$$\begin{aligned}
\langle K^{a_1}(x_1) K^{a_2}(x_2) K^{a_3}(x_3) \rangle_0 &= \frac{\frac{1}{2} k_G I_0 f^{a_1 a_2 a_3}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \\
\langle K^{a_1}(x_1) K^{a_2}(x_2) I \rangle_0 &= \frac{\frac{1}{2} k_G I_0^2 \delta^{a_1 a_2}}{(x_1 - x_2)^2} \\
\langle III \rangle_0 &= I_0^3\,.
\end{aligned} \tag{5.11}$$

A direct calculation of the correlators (5.11) using worldsheet techniques gives

$$\begin{aligned}
\langle K^{a_1}(x_1) K^{a_2}(x_2) K^{a_3}(x_3) \rangle_0 &= \frac{\frac{1}{2} k_G \frac{C_{111}}{k^2} f^{a_1 a_2 a_3}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \\
\langle K^{a_1}(x_1) K^{a_2}(x_2) I \rangle_0 &= \frac{\frac{1}{2} k_G \frac{C_{111}}{k^3} \delta^{a_1 a_2}}{(x_1 - x_2)^2} \\
\langle III \rangle_0 &= \frac{C_{111}}{k^4},
\end{aligned} \tag{5.12}$$

where C_{111} is defined by

$$\langle \Phi_1(x_1, \bar{x}_1; z_1, \bar{z}_1) \Phi_1(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_1(x_3, \bar{x}_3; z_3, \bar{z}_3) \rangle_0 = \frac{C_{111}}{|x_1 - x_2|^2 |x_1 - x_3|^2 |x_2 - x_3|^2}\,.$$

Comparing (5.11) and (5.12) we find three equations for the two unknowns C_{111} and I_0. The fact that a solution exists,

$$C_{111} = k\,, \qquad I_0 = \frac{1}{k}\,, \tag{5.13}$$

provides a consistency check of the assumption (5.10).

[15]The spherical correlation functions in the next few equations are divided by the partition sum Z_0, so that $\langle 1 \rangle_0 = 1$.

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The leading order contribution to the spacetime level (5.4) is thus $k_G Z_0 I_0$. The partition sum on the sphere can presumably be calculated using worldsheet methods; this has not been done to date and we will not attempt to compute it here. Alternatively, one can use the fact that the partition sum of any string theory on the sphere is equal to the classical action evaluated on the solution of the equations of motion. Substituting (1.4) into (1.3) gives zero, but the action has a finite surface term of order kp. Therefore, $Z_0 \simeq kp$ up to a numerical coefficient (independent of k, p) which we will not attempt to fix. This leads to

$$k_G^{(st)} \approx k_G Z_0 I_0 = k_G p \tag{5.14}$$

(where the $\approx$ sign refers to the large p approximation) in agreement with the spacetime result (1.9). The numerical coefficient, which has not been computed, can be fixed either by a calculation of the worldsheet partition sum or by a Gibbons-Hawking type calculation of the classical action corresponding to (1.3), (1.4); the latter is known to give the correct answer (5.14).

6. The spacetime Virasoro symmetry

In this section we generalize the results of the previous sections to the case of the spacetime Virasoro symmetry. The discussion is analogous to the one for affine Lie algebras, so we mention only the new aspects. Using the expected spacetime scaling dimension of the spacetime stress tensor $T(x)$ we can write it as a linear combination

$$T(x) = \int d^2z \left[A_1 J(x;z)\partial_x^2\Phi_1 + A_2\partial_x J\partial_x\Phi_1 + A_3\partial_x^2 J\Phi_1 \right] \bar{J}(\bar{x};\bar{z}) \,. \tag{6.1}$$

BRST invariance imposes a constraint on the coefficients A_i. One solution, with the overall normalization fixed below, is

$$\begin{aligned} T(x) &= \frac{1}{2k}\int d^2z \left[J(x;z)\partial_x^2\Phi_1 + 3\partial_x J\partial_x\Phi_1 + 3\partial_x^2 J\Phi_1 \right] \bar{J}(\bar{x};\bar{z}) \\ &= \frac{1}{2}\oint \frac{dz}{2\pi i}\left[J(x;z)\partial_x^2\Lambda + 3\partial_x J\partial_x\Lambda + 3\Lambda\partial_x^2 J \right] . \end{aligned} \tag{6.2}$$

We have already seen above that $\partial_x \int d^2z J\bar{J}\Phi_1 = \int d^2z\partial_z(\bar{J}\Phi_1) = 0$. Therefore, we have the freedom to shift T by any constant times $\int d^2z\partial_x^2(J\Phi_1)\bar{J}$ (this operator is BRST exact and vanishes in correlation functions). Using this freedom we can write

$$T(x) = \frac{1}{2k}\int d^2z \left(\partial_x J\partial_x\Phi_1 + 2\partial_x^2 J\Phi_1\right)\bar{J}(\bar{x};\bar{z}) = \frac{1}{2}\oint\frac{dz}{2\pi i}\left(\partial_x J\partial_x\Lambda + 2\Lambda\partial_x^2 J\right) . \tag{6.3}$$

Note that:

1. $T(x)$ is a tensor of weight $(2,0)$ in spacetime. In fact, imposing that it should be such a tensor is an alternative way of fixing the relative coefficients in (6.1).
2. One might wonder whether it is possible to generate tensors of weight $(h,0)$ with $h>2$ in string theory on AdS_3 by generalizing the logic of (6.1)–(6.3). It is easy to see that there are no new operators (in addition to $\partial_x^n T(x)$) at higher levels; the reason is that $\partial_x^3 J(x;z)=0$.

It is usually said that in string theory on AdS_3 (with B_{NS} turned on) the dilaton is a fixed scalar, meaning that it is massive. These statements refer *not* to the vertex operators (4.18), (6.3), which are almost pure gauge. The vertex operator of the massive dilaton is

$$D(x,\bar{x})=\int d^2z\left(\partial_x J\partial_x+2\partial_x^2 J\right)\left(\partial_{\bar{x}}\bar{J}\partial_{\bar{x}}+2\partial_{\bar{x}}^2\bar{J}\right)\Phi_1\,, \tag{6.4}$$

which has spacetime scaling dimension $(2,2)$ and is only quasi-primary under the spacetime Virasoro algebra – it has the quantum numbers of $T(x)\bar{T}(\bar{x})$. A similar statement can be made regarding the gauge bosons associated with a worldsheet affine Lie algebra $k^a(z)$. The gluons get a mass from the Chern-Simons term discussed in section 1. The corresponding vertex operators are

$$A^a(x,\bar{x})=\int d^2z k^a(z)\left(\partial_{\bar{x}}\bar{J}\partial_{\bar{x}}\Phi_1+2\partial_{\bar{x}}^2\bar{J}\Phi_1\right). \tag{6.5}$$

The operators A^a have scaling dimension $(1,2)$ in the spacetime CFT and have the quantum numbers of $K^a(x)\bar{T}(\bar{x})$.

However, $D(x,\bar{x})\neq T(x)\bar{T}(\bar{x})$ because the two point function of $D(x,\bar{x})$ scales like $1/g_s^2$, while that of $T\bar{T}$ arises from a disconnected diagram and scales like $1/g_s^4$. Similarly, $A^a(x,\bar{x})\neq K^a(x)\bar{T}(\bar{x})$.

To derive the Virasoro Ward identity it is convenient to insert into correlators the operator $\partial_{\bar{x}}T$,

$$\partial_{\bar{x}}T(x)=\frac{1}{2}\oint\frac{dz}{2i}\left[\partial_x J\partial_x\Phi_1+2\Phi_1\partial_x^2 J\right]. \tag{6.6}$$

A calculation of the sort outlined above for the affine Lie algebra case can now be performed for the stress tensor and one finds the correct Ward identities for TT and TW_h,

$$\begin{aligned}T(x)T(y)&\sim\frac{\frac{1}{2}c^{(st)}}{(x-y)^4}+\frac{2T(y)}{(x-y)^2}+\frac{\partial_y T}{x-y}\\ T(x)W_h(s,\bar{s})&\sim\frac{hW_h(s,\bar{s})}{(x-s)^2}+\frac{\partial_s W_h}{x-s}\end{aligned} \tag{6.7}$$

with $c^{(st)}=6kP(g_s)$.

7. Relation to [4]

The original analysis of [4] is valid in the free field region of $\phi \to \infty$. Therefore, in order to compare our exact discussion above with [4] we should first consider the behavior of all our operators in that limit.

Using $\lim_{\phi\to\infty} \Lambda = 1/(x-\gamma)$ (2.18) in the expression for the current (3.6) we find

$$K_f^a(x) \equiv \lim_{\phi\to\infty} K^a(x) = -\frac{1}{\pi}\int_{\Sigma'} d^2z k^a(z)\partial_{\bar{z}}\frac{1}{x-\gamma(z)}\,, \tag{7.1}$$

or after performing the integration by parts (4.4), (4.5)

$$K_f^a(x) = \sum_i \oint_{C_i} \frac{dz}{2\pi i}\frac{k^a(z)}{x-\gamma(z)}\,. \tag{7.2}$$

Here we can see the relation to [4] by expanding (7.2) in powers of x/γ or γ/x; i.e. performing the Fourier transform discussed in section 2, to find

$$K_n^a = \oint \frac{dz}{2\pi i} k^a(z)\gamma^n\,. \tag{7.3}$$

Recalling that $\gamma(z)$ is holomorphic in the $\phi \to \infty$ limit, we can rewrite (7.1) as

$$K_f^a(x) = \int_{\Sigma'} d^2z k^a(z)\partial_{\bar{z}}\bar{\gamma}\delta^2(x-\gamma(z))\,. \tag{7.4}$$

Using

$$\lim_{\phi\to\infty} \Phi_1 = \delta^2(\gamma - x) \tag{7.5}$$

and the expression for $\bar{J}$, (2.12), we recognize (7.4) as the limit as $\phi \to \infty$ of (3.7).

Similarly, using (7.5) and (2.12) in the exact expression for I (4.18) we find

$$\begin{aligned} I_f \equiv \lim_{\phi\to\infty} I &= \int_{\Sigma'} d^2z \delta^2(x-\gamma)\partial\gamma\bar{\partial}\bar{\gamma} \\ &= -\frac{1}{\pi}\int_{\Sigma'} d^2z \partial_z\gamma\partial_{\bar{z}}\frac{1}{x-\gamma(z)} = \sum_i \oint_{C_i}\frac{dz}{2\pi i}\frac{\partial_z\gamma}{x-\gamma(z)}\,. \end{aligned} \tag{7.6}$$

Expanding this in powers of x/γ we find the Fourier modes

$$I_n = \oint \frac{\partial_z\gamma}{\gamma}\left(\frac{x}{\gamma}\right)^n = \delta_{n,0}\oint\frac{\partial_z\gamma}{\gamma} \tag{7.7}$$

in agreement with the corresponding expression in [4].

The analysis of the Ward identities of the previous sections simplifies considerably in the $\phi \to \infty$ limit and becomes identical to [4] (up to the Fourier transform from x space to the modes). Indeed, consider the $\phi \to \infty$ limit of the correlation function (4.1)

$$\langle K_f^a(x)K_f^b(y)W_h(s,\bar{s})\cdots\rangle\,, \tag{7.8}$$

with operators $W_h(s)$, which are proportional to $e^{2(h-1)\phi}\delta^2(\gamma - x)$ (see, (2.8)). (In [4] the Fourier modes of these operators (2.42) were used.) The contribution to (7.8) from a contour C around one of the W_h operators is

$$K_f^a(x)W_h(s) \sim \int d^2w \oint_w \frac{dz}{2\pi i}\frac{k^a(z)}{x-\gamma(z)}W_h(s,\bar{s};w,\bar{w}) = \frac{t^a(R)}{x-y}\int d^2w W_h(s,\bar{s};w,\bar{w}) \,, \tag{7.9}$$

where we used the OPE (4.2) and the fact that W_h is proportional to a delta function. We also used the fact that there is no short distance singularity in the OPE $\gamma(z)\gamma(w)$ (again in the free field region $\phi \to \infty$).

A similar calculation of the contribution from the region where K_f^a approaches on the worldsheet another current K_f^b is most conveniently done by using the representation (7.2) for K_f^a and (7.4) for K_f^b. There are now two contributions coming from the single and double poles in the worldsheet OPE (3.1). They lead to

$$\begin{aligned} K_f^a(x)K_f^b(y) &\sim \int d^2w \oint_w \frac{dz}{2\pi i}\frac{k^a(z)}{x-\gamma(z)}k^b(w)\partial_{\bar{w}}\bar{\gamma}\delta^2(y-\gamma(w)) \\ &= \frac{f^{abc}}{x-y}\int d^2w k^c(w)\partial_{\bar{w}}\bar{\gamma}\delta^2(y-\gamma(w)) + \frac{\frac{1}{2}k_G\delta^{ab}I_f}{(x-y)^2} \,. \end{aligned} \tag{7.10}$$

Thus, (7.9), (7.10) lead to the spacetime Ward identities

$$\begin{aligned} K_f^a(x)W_h(s) &\sim \frac{t^a(R)}{x-y}W_h(s) \\ K_f^a(x)K_f^b(y) &\sim \frac{f^{abc}}{x-y}K_f^c(y) + \frac{\frac{1}{2}I_f k_G\delta^{ab}}{(x-y)^2} \,, \end{aligned} \tag{7.11}$$

which are the $\phi \to \infty$ analogs of (4.3), (4.19). The Virasoro generator $T(x)$ constructed in section 6 also simplifies in the limit $\phi \to \infty$. Using (2.18) in (6.3) one finds

$$T(x) = \frac{1}{2}\oint\frac{dz}{2\pi i}\left[-\frac{\partial_x J}{(x-\gamma)^2} + \frac{2\partial_x^2 J}{x-\gamma}\right] . \tag{7.12}$$

Since (2.11) $\partial_x J = 2J^3 - 2xJ^-$, $\partial_x^2 J = -2J^-$, expanding in powers of γ/x leads to

$$-L_n = \oint\frac{dz}{2\pi i}\left[(n+1)J^3\gamma^n - nJ^-\gamma^{n+1}\right] \tag{7.13}$$

in agreement with [4]. The OPE algebra (6.7) can be derived similarly to the affine Lie algebra case discussed above; as there, the derivation is related to the discussion of [4] by a Fourier transform.

So far we have shown how the exact expressions in the previous sections simplify as $\phi \to \infty$ and reproduce the (Fourier transform to x space of the) corresponding expressions in [4]. We next look more closely at the relation between the two calculations.

The vertex operators in the theory can be determined in the free field region $\phi \to \infty$. The quantum numbers of these vertex operators can also be determined in the asymptotic region. However, their correlation functions depend on the entire space and cannot be computed using only the free field region. An exception to this is Ward identities like (7.11). These are associated with singularities in x space. As explained in the introduction, such singularities are determined by the $\phi \to \infty$ limit. For holomorphic operators like $K^a(x)$ the correlation functions are completely determined by the singularities and therefore the asymptotic analysis suffices. The rest of the information in the correlation functions depends on the full structure of the theory including the bulk of the target space where ϕ is small. We conclude that the asymptotic expressions of the vertex operators, as in [4], are good for some purposes like enumerating the vertex operators in the theory and for deriving Ward identities like (7.11), but in order to find the exact correlation functions the more complete form of the vertex operators given in this paper is necessary.

Another important point has to be discussed. The expression (7.6) for the asymptotic form of I is the degree of the holomorphic map $\gamma(z)$. Clearly it is an integer q independent of the other insertions in the correlation function

$$\langle I_f V_1 \ldots V_n \rangle = q \langle V_1 \ldots V_n \rangle \,, \tag{7.14}$$

and therefore I_f is central. For example, if $\gamma(z) = z$, $q = 1$. Other values of q can be obtained, e.g. from $\gamma(z) = z^q$. Even though I_f is simply the limit of I as $\phi \to \infty$, the expression for the central charge (7.6) differs from the discussion in section 5 in several crucial respects:

1. In section 5 the leading order contribution to the central charge came, as in [5], from a disconnected diagram, while here the contribution is from a connected diagram.

2. In section 5, $p = \langle I \rangle_0$ was not quantized. This is consistent with the expectations from the spacetime analysis described in the introduction that p is quantized only non-perturbatively. Here, $q = \langle I_f \rangle$ is quantized already classically.

3. In the discussion in section 5 connected diagrams also contribute to $P(g_s)$. As we saw in (5.13), the leading order connected contribution is $I_0 = 1/k$ which is not even quantized. This is consistent since there is another (disconnected) contribution at the same order. In the computation with I_f the entire contribution comes from the connected diagram.

4. The expression (7.6) is non-zero only if the worldsheet is mapped to an infinite plane in H_3^+. Maps of the worldsheet to a compact region in the target space lead to $\langle I_f \rangle = 0$. On the other hand, we expect that, at least semiclassically, the leading contribution to the functional integral comes from such maps. Indeed, the expressions in section 5 can come from maps to a compact region.

What is then the meaning of $\langle I_f \rangle = q$? Clearly, it receives contributions only from worldsheets at large ϕ, which have the topology of a plane rather than from small worldsheets. Unlike the discussion in section 5, which applies to short strings propagating in the bulk of AdS_3, the discussion of the central charge here and in [4] applies to a worldsheet of a long string in the region $\phi \to \infty$.

Such long strings appeared in [16] and a detailed discussion of them appears in [17]. Recall that p is the number of fundamental strings creating our background. Therefore, q coincident long strings act as a domain wall changing the value of p. On one of its sides we find the original value of p. On the other side the value of p is screened to $p - q$. The effective theory of the long string is a conformal field theory. The vertex operators in this theory describe the emission and absorption of short strings from it. However, since the worldsheet of the long string is at large ϕ, we can safely use the asymptotic form of the vertex operators. The worldsheet in this problem is the worldsheet of the long string which is non-compact. Furthermore, in the effective theory of the long string there are no disconnected worldsheets, as these describe short strings only. $\langle I_f \rangle$ measures the number of long strings. Therefore it is quantized even at tree level.

This interpretation of the calculation of $\langle I_f \rangle = q$ neatly explains the four puzzles above.

To summarize, the structure emerging from [4, 17] and this note is the following. String theory on AdS_3 with a given value of p (1.5) contains many different sectors. One sector corresponds to "short strings", i.e. maps from the worldsheet of the string to a compact region in AdS_3. This is the sector described in this note. In it, p determines the fundamental string coupling, $g_s^2 \propto 1/p$. The quantization of g_s is a non-perturbative effect which is not seen in our classical analysis, as explained in section 1.

Other sectors contain in addition $0 < q \leq p$ "long strings" at infinity. These sectors contain two kinds of excitations, fundamental strings propagating in the bulk of AdS_3 and small ripples on the long strings at infinity.

The analysis of [4] plays two roles in string theory on AdS_3. In the short string sector of the theory, physical observables correspond to "microscopic operators" in the language of Liouville theory [25]; their wavefunctions are exponentially supported at the boundary $\phi \to \infty$. Thus in order to identify the observables and study their transformation properties under the spacetime symmetries, it is enough to analyze the wavefunctions at large ϕ; this is done in [4]. In particular, the analysis of [4] proves that fundamental string states transform in representations of the infinite spacetime symmetry algebra; it also exhibits the set of representations that appear in the fundamental string spectrum.

Correlation functions of these observables (and in particular the central charge) are not supported at $\phi \to \infty$, and thus one needs to analyze the behavior of the theory in the bulk of AdS_3.

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A second place where the analysis of [4] is applicable is in studying the effective theory of the q long strings located near the boundary of AdS_3. There, the free field description is applicable. The vertex operators of [4] describe small excitations of the long strings and the current algebra generators (e.g. (7.1)) can be used to study the transformation properties of these excitations under the spacetime symmetry algebra. I_f (7.6) keeps track of the central charge carried by these long strings. As one would expect, it is quantized at tree level.

8. Superstrings on $AdS_3 \times S^3 \times T^4$

The supersymmetric case adds to the previous discussion a few new issues, such as the spacetime physics of RR sector fields. In this section we briefly comment on the generalization of the construction of the previous sections to the superstring (see [4, 7] for more details).

In addition to the bosonic sector of the worldsheet theory we now have free worldsheet fermions $\psi^A(z)$ which transform in the adjoint of SL(2) (and a similar structure for the other worldsheet chirality). The total SL(2) currents $J^A(z)$ now have two contributions: a level $k+2$ bosonic current J_B^A, and a level -2 fermionic current J_F^A; $J^A = J_B^A + J_F^A$.

It is convenient to follow the discussion of the bosonic case and use (2.9) to define

$$\psi(x;z) = 2x\psi^3 - \psi^+ - x^2\psi^- \,. \tag{8.1}$$

$\psi(x;z)$ satisfy the OPE algebra[16]

$$\psi(x;z)\psi(y;w) \sim \frac{2(x-y)^2}{z-w} \,. \tag{8.2}$$

Note that $\psi(x;z)$, like $J(x;z)$ in the bosonic case, has spacetime scaling dimension -1. The fermionic SL(2) current can be written in terms of $\psi(x;z)$ as

$$J_F(x;z) = \frac{1}{2}\psi(x;z)\partial_x\psi(x;z) \,. \tag{8.3}$$

It is easy to check that it satisfies (2.24) with $k=-2$. Thus, the total SL(2) current is

$$J(x;z) = J_B(x;z) + J_F(x;z) \,. \tag{8.4}$$

The worldsheet stress tensor (2.26) of the bosonic degrees of freedom is given by

$$T_B^{\rm ws} = \frac{1}{2k}\left[J_B\partial_x^2 J_B - \frac{1}{2}(\partial_x J_B)^2\right] . \tag{8.5}$$

[16]We normalize ψ^A as: $\psi^A(z)\psi^B(w) = \eta^{AB}/(z-w)$.

the fermionic worldsheet stress tensor is

$$T_F^{\rm ws} = -\frac{1}{2}\psi^A\partial_x\psi_A = \frac{1}{8}\left(\partial_x\psi\partial_z\partial_x\psi - \partial_x^2\psi\partial_z\psi - \psi\partial_z\partial_x^2\psi\right), \tag{8.6}$$

and the superconformal generator is (up to an overall constant which will not play a role below)

$$G^{\rm ws}(z) = -\partial_x\psi\partial_x J_B + \partial_x^2\psi J_B + \psi\partial_x^2 J_B + \frac{1}{2}\psi\partial_x\psi\partial_x^2\psi\,. \tag{8.7}$$

As in the bosonic case, it is easy to see that $\partial_x T_F^{\rm ws} = \partial_x G^{\rm ws} = 0$, and that (8.6), (8.7) transform as tensors of weight $(0,0)$ in spacetime.

The chiral GSO projection implies that the worldsheet theory contains $(\Delta, \bar\Delta) = (1,0)$ holomorphic operators

$$\theta_r^{\alpha\mu} = e^{-\frac{1}{2}\varphi}S_r^{\alpha\mu}, \tag{8.8}$$

where φ is the bosonized ghost of the fermionic string. The operators θ have $h = -1/2$ and transform in the $(2,2)$ of the $\mathrm{SO}(3)\times\mathrm{SO}(4)$ symmetry acting on the worldsheet fermions $(\psi^A, \chi^a, \lambda^i)$ (see [4] for notation). $r = \pm 1/2$ is the SL(2) index; $\alpha = \pm$ is the SU(2) index. μ denotes a spinor of SO(4); the other spinor of SO(4) will be denoted by $\widetilde\mu$. There is a second set of holomorphic operators $\widetilde\theta_r^{\alpha\widetilde\mu}$ which will be useful below.

As shown in [4], $\oint dz\theta_r^{\alpha\mu}$ generate the global $N=4$ superconformal algebra. To construct the local superconformal currents it is convenient to define the x dependent spin field

$$\theta^{\alpha\mu}(x) = e^{-xJ_0^-}\theta_{\frac{1}{2}}^{\alpha\mu}e^{xJ_0^-} = \theta_{\frac{1}{2}}^{\alpha\mu} - x\theta_{-\frac{1}{2}}^{\alpha\mu}\,. \tag{8.9}$$

The OPE of $\psi(x;z)$ with $\theta^{\alpha\mu}$ is

$$\begin{aligned}\psi(x;z)\theta^{\alpha\mu}(y;w) &\sim \frac{x-y}{(z-w)^{\frac{1}{2}}}\theta^{\alpha\mu}(x;w)\\ &= \frac{1}{(z-w)^{\frac{1}{2}}}\left[(x-y)^2\partial_y + (x-y)\right]\theta^{\alpha\mu}(y;w)\,.\end{aligned} \tag{8.10}$$

Furthermore, one can show that

$$\theta^{\alpha\mu}(x;z)\theta^{\beta\nu}(y;w) \sim \frac{\delta^{\mu\nu}e^{-\varphi}}{z-w}\left[\delta^{\alpha\beta}\psi(x;w) + (x-y)\sigma_a^{\alpha\beta}\chi^a(w)\right]. \tag{8.11}$$

Following the by now familiar pattern, we are looking for spacetime holomorphic operators involving the worldsheet current $\theta^{\alpha\mu}$. One can check (using (8.5)–(8.7)) that the operator

$$\Psi^{\alpha\widetilde\mu}(x) = -\frac{1}{\pi k}\int d^2z\widetilde\theta^{\alpha\widetilde\mu}(x;z)e^{-\bar\varphi}\bar\psi(\bar x;\bar z)\Phi_1(x,\bar x;z,\bar z) \tag{8.12}$$

is BRST invariant. It corresponds to a dimension $(\frac{1}{2},0)$ fermionic operator. In fact, the four spacetime fermions $\Psi^{\widetilde{\alpha\mu}}(x)$ are the bottom components of $N=4$ superfields whose top components are the $\mathrm{U}(1)^4$ currents

$$P^i(x) = -\frac{1}{k\pi}\int d^2z e^{-\varphi-\bar{\varphi}}\lambda^i(z)\bar{\psi}(\bar{x};\bar{z})\Phi_1(x,\bar{x};z,\bar{z})\,, \tag{8.13}$$

which were constructed in [4] and further discussed in [7].

At the next level we find the spacetime $N=4$ superconformal currents

$$G^{\alpha\mu}(x) = -\frac{1}{\pi k}\int d^2z\,(\theta^{\alpha\mu}\partial_x\Phi_1 + 2\Phi_1\partial_x\theta^{\alpha\mu})\,e^{-\bar{\varphi}}\bar{\psi}. \tag{8.14}$$

The form of the stress tensor is an obvious generalization of (6.2), (6.3). Using the formalism of the previous sections one can show that the superconformal generators satisfy the $N=4$ superconformal algebra in spacetime.

Note that just as in the bosonic case studied in the previous sections, the operators (8.12–(8.14) describing holomorphic operators in the spacetime CFT are "almost pure gauge." The basic observation[17] is that the $h=0$ combination $\exp(-\varphi)\psi(x;z)\Phi_1(x,\bar{x};z,\bar{z})$ can be written as

$$e^{-\varphi}\psi(x;z)\Phi_1(x,\bar{x};z,\bar{z}) = \{Q_{BRST},\partial\xi e^{-2\varphi}\Lambda(x,\bar{x};z,\bar{z})\}\,, \tag{8.15}$$

where ξ is a field arising in the bosonization of the bosonic β, γ ghosts in the fermionic string, and Λ is our old friend (2.16). The fact that Λ is not a good observable in CFT on AdS_3 again implies that despite (8.15), $\exp(-\varphi)\psi(x;z)\Phi_1(x,\bar{x};z,\bar{z})$ is not BRST exact. At the same time its correlation functions can be computed by a straightforward generalization of the bosonic analysis. This can be done by working directly with the -1 picture field (8.15), or by picture changing it to the 0 picture, where it can be written as a total derivative, $J_B(x;z)\Phi_1(x,\bar{x};z,\bar{z}) \sim \partial_{\bar{z}}\Lambda$, and one can proceed as in the previous sections.

In section 1 we saw from a target space analysis that RR gauge fields on AdS_3 have a mass of the same order of magnitude $(1/\sqrt{k})$ as that of the NS sector gauge fields. This is manifest in the worldsheet formalism. The j'th partial wave of an NS gauge field obtained by Kaluza-Klein reduction from ten to three dimensions is described by the vertex operator

$$A^i_j = e^{-\varphi}\psi(x;z)e^{-\bar{\varphi}}\bar{\lambda}^i(\bar{z})\Phi_{j+1}(x,\bar{x})V_j(z,\bar{z})\,, \tag{8.16}$$

where V_j is the wavefunction on S^3 (see [4, 7] for details). A_j transforms in the spin (j,j) representation of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ and has spacetime scaling dimension

$$(h,\bar{h}) = (j,j+1)\,. \tag{8.17}$$

[17]In (8.12)–(8.14) we actually need the antiholomorphic version of the formulae below.

The corresponding RR gauge field is described by the vertex operator

$$R_j^{\widetilde{\mu}\bar{\mu}} = e^{-\frac{\varphi}{2}-\frac{\bar{\varphi}}{2}} \widetilde{\theta}^{\alpha\widetilde{\mu}} \left(\bar{\theta}^{\bar{\alpha}\bar{\mu}} \partial_{\bar{x}} \Phi_{j+\frac{1}{2}} + 2\Phi_{j+\frac{1}{2}} \partial_{\bar{x}} \bar{\theta}^{\bar{\alpha}\bar{\mu}} \right) V_{j-\frac{1}{2}}\,, \tag{8.18}$$

where $\widetilde{\theta}$, $\bar{\theta}$ and V are coupled into a representation with total spin (j, j) under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ (as (8.16)). The spacetime scaling dimension of (8.18) is again given by (8.17). In fact, the operators (A_j, R_j) together with their antiholomorphic counterparts form multiplets of the U-duality group. These multiplets were studied in [7].

In section 1 we also saw that string theory on $AdS_3 \times S^3 \times T^4$ is expected to have in addition to the NS $\mathrm{U}(1)_L^4 \times \mathrm{U}(1)_R^4$ current algebra described on the worldsheet by (8.13) and its right moving analog, a $\mathrm{U}(1)_L^4 \times \mathrm{U}(1)_R^4$ current algebra associated with RR gauge fields (see the discussion after (1.10)). One would expect this current algebra to correspond to RR vertex operators on the worldsheet. We have been unable to find vertex operators with the right properties. It is possible that we have not looked carefully enough. However, since the spacetime CFT is two dimensional and there are no perturbative string states which are charged under this RR current algebra, it forms a decoupled sector of the theory. Therefore, it is not clear whether such vertex operators should be found in our worldsheet approach.

9. Summary

In this paper we studied string theory in an AdS_3 background. Most of the discussion did not depend on many of the details of how this background is generated. On general grounds, the three dimensional low energy effective lagrangean is (1.3)

$$e^{-2\phi}\sqrt{-g}\left(\frac{4}{k} + R + 4(\nabla\phi)^2 - \frac{1}{12}H_{\mu\nu\rho}^2\right). \tag{9.1}$$

The equations of motion show that H is nonzero and the zero mode of ϕ is an arbitrary integration constant. The nonzero H can be interpreted as arising from a source of strength $p = 4\pi/\hbar\sqrt{k}$ at infinity. Classically and to all orders in perturbation theory p is an arbitrary real number. When there is a worldsheet current algebra $\widehat{G}$ at level k_G, there are gauge fields in spacetime and the lagrangean (9.1) is modified to (1.6). As a result, $\widehat{G}$ current algebra at level $k_G^{(st)} = k_G p$ appears in the spectrum. Hence, nonperturbatively p must be quantized.

We then studied various properties of the worldsheet CFT on AdS_3. Following [22] we parameterized the operators in terms of their SL(2) quantum number h and a continuous complex parameter x, which is interpreted as a coordinate on the boundary of the space, on which the spacetime CFT "lives." The operator (2.7)

$$\Phi_1 = \frac{1}{\pi}\left(\frac{1}{(\gamma - x)(\bar{\gamma} - \bar{x})e^{\phi} + e^{-\phi}}\right)^2, \tag{9.2}$$

which has worldsheet conformal dimensions $(0,0)$ and spacetime dimensions $(1,1)$ plays an important role in the theory. Using this operator, the worldsheet SL(2) current $J(x;z)$ and $\widehat{G}$ worldsheet currents k^a, we constructed the vertex operators of the spacetime current algebra (3.5), the spacetime identity operator (4.18), and spacetime Virasoro algebra (6.2)

$$\begin{aligned} K^a(x) &= -\frac{1}{k}\int d^2z k^a(z)\bar{J}(\bar{x};\bar{z})\Phi_1(x,\bar{x};z,\bar{z}) \\ I &= \frac{1}{k^2}\int d^2z J(x;z)\bar{J}(\bar{x};\bar{z})\Phi_1(x,\bar{x};z,\bar{z}) \\ T(x) &= \frac{1}{2k}\int d^2z \left(\partial_x J\partial_x\Phi_1 + 2\partial_x^2 J\Phi_1\right)\bar{J}(\bar{x};\bar{z})\,. \end{aligned} \tag{9.3}$$

Assuming (2.35)

$$\lim_{z_1\to z_2}\Phi_1(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_h(x_2,\bar{x}_2;z_2,\bar{z}_2) = \delta^2(x_1-x_2)\Phi_h(x_2,\bar{x}_2;z_2,\bar{z}_2)\,, \tag{9.4}$$

which can be proven in the leading order in the large k semiclassical expansion, we showed that (9.3) satisfy the correct operator product expansion in spacetime

$$\begin{aligned} K^a(x)K^b(y) &\sim \frac{\frac{1}{2}k_G^{(st)}\delta^{ab}}{(x-y)^2} + \frac{f^{abc}K^c(y)}{x-y} \\ K^a(x)W_h(y,\bar{y}) &\sim \frac{t^a(R)W_h(y,\bar{y})}{x-y} \\ T(x)T(y) &\sim \frac{\frac{1}{2}c^{(st)}}{(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial_y T}{x-y} \\ T(x)W_h(y,\bar{y}) &\sim \frac{hW_h(y,\bar{y})}{(x-y)^2} + \frac{\partial_y W_h}{x-y}\,. \end{aligned} \tag{9.5}$$

The fact that I in (9.3) behaves like a spacetime identity operator follows from another assumption which we did not prove, (5.7).

As we stressed, since the first quantized string description is in Landau gauge, we do not have the freedom to gauge the operators (9.3) to the boundary. Therefore the corresponding vertex operators have nontrivial profiles in the bulk of AdS_3.

Extending these techniques to the superstring we found the expected $N=4$ spacetime superconformal algebra.

As $\phi\to\infty$, one recovers the results of [4]. The discussion of [4] can be used for two distinct purposes in string theory on AdS_3. In the "short string" sector it is useful for enumerating the observables and studying their transformation properties under the spacetime symmetry algebra. It can also be used to study the physics of long strings in AdS_3. One important difference between the physics of short strings and long strings is the value of the central charge. For short strings the central charge arises (in leading order), as in [5], from disconnected diagrams. As expected from the spacetime analysis, it is not quantized in perturbation theory. For long strings the central charge arises as in [4] from a cylindrical connected diagram. It is quantized at the leading order.

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